Notes on some past exam questions (relevant to the topics discussed in the rev. lecture)

June 2013 - Q10

- Derivation of $\bar{n}_i = \dfrac{1}{e^{\beta(\varepsilon_i - \mu)} - 1}$

  — See Notes pp 169-171

- Derivation of $\dfrac{N}{V} = \dfrac{(2m)^{3/2}}{4\pi^2\hbar^3} \displaystyle\int_0^\infty \frac{\varepsilon^{1/2}\, d\varepsilon}{e^{\beta(\varepsilon-\mu)} - 1}$

  — just did this in Rev. Lecture, p. 205
  (the question assumes $2S+1 = 2$ implicitly)

- Chemical potential goes to 0 from below (p. 209)
- Derivation of T_c without computing the integral: p. 209
- # of particles in the ground state:

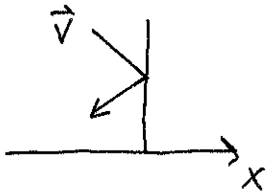
  $\bar{n}_0 = N - N_{exc} = N\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right]$ (p. 210)

- Bose condensate — these $\bar{n}_0$ particles in the ground state, free to "evaporate" into N_{exc} as T goes up or to acquire more particles from amongst N_{exc} as T goes down.

June 2013 - Q6

Pressure in 2D : same calculation as on p. 201:

$$p = \int d\Phi(\vec{v}) \cdot 2mv_x = \int_{v_x>0} 2mv_x \cdot \underbrace{v_x\, n f(v) d^2\vec{v}}_{\text{flux in 2D}} =$$



$$= \int_{\text{all } v_x} m v_x^2\, n f(v) d^2\vec{v} = \langle m n v_x^2 \rangle =$$

because $f(-v_x) = f(v_x)$ isotropy

isotropy in 2D

$$= \frac{1}{2} m n \langle v^2 \rangle$$

$\langle v_x^2 \rangle + \langle v_y^2 \rangle = \langle v^2 \rangle$

and $\langle v_x^2 \rangle = \langle v_y^2 \rangle$

answer is requested "in terms of an appropriate average over the velocity distribution"

June 2012 - Q10

- Density of states in 2D electron gas ($S = 1/2$)
  - derived on p. 205 (g is D)

$$D(\varepsilon) = \frac{Am}{\pi\hbar^2} \quad (\text{at } 2S+1 = 2)$$

- # of electrons in $[\varepsilon, \varepsilon + d\varepsilon]$:

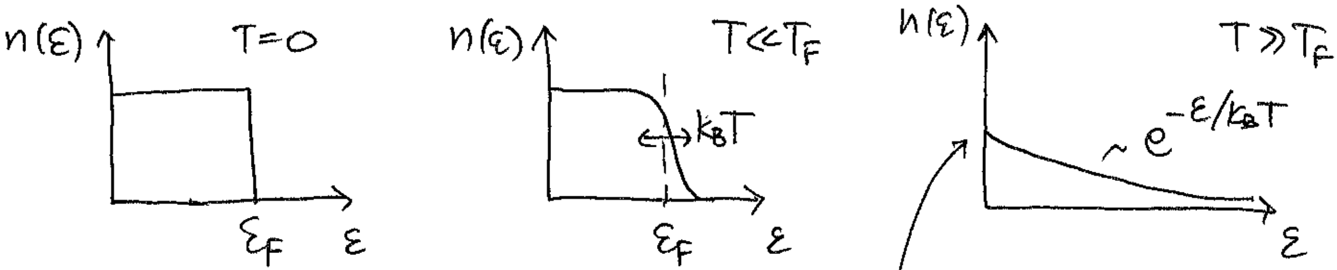
$$n(\varepsilon)\,d\varepsilon = D(\varepsilon) \cdot \frac{1}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon$$

$\beta = 1/k_B T$ (T: temperature)

μ is ch. potential

- Fermi temperature: $T_F = \frac{\varepsilon_F}{k_B}$,

ε_F is the energy of the highest occupied level at $T=0$



NB: $n(0)$ in this regime $\ll$ than $n(0)$ for $T=0$! because $\int n(\varepsilon)\,d\varepsilon = N$ fixed.

- Mean energy per particle at $T=0$: p.206
- Heat capacity calculation w/o numerical prefactors: p.208

June 2010 - Q4

Flux of kinetic energy through effusion hole: as on pp201-202,

$$J = \int_{v_z>0} d\Phi(\vec{v}) \cdot \frac{mv^2}{2} =$$

$$= \frac{nm}{2}\int_0^\infty dv\, v^5 f(v) \underbrace{\int_0^{\pi/2} d\theta \cos\theta \sin\theta}_{=\frac{1}{2}} \underbrace{\int_0^{2\pi} d\varphi}_{=2\pi}$$

in the formulation of the question, this is already included into $F(v,\theta)$

$$= \frac{mn}{2}\pi \int_0^\infty dv\, v^5 \frac{e^{-v^2/v_{th}^2}}{(\pi v_{th}^2)^{3/2}}$$

$$= \frac{mn}{2\sqrt{\pi}\, v_{th}^3} \underbrace{\int_0^\infty dv\, v^5 e^{-\alpha v^2}}_{x=v^2} = \frac{mn}{2\sqrt{\pi}} v_{th}^3 = \frac{mn}{2\sqrt{\pi}}\left(\frac{2k_BT}{m}\right)^{3/2}$$

$$\frac{1}{2}\int_0^\infty dx\, x^2 e^{-\alpha x} = \frac{1}{2}\frac{\partial^2}{\partial\alpha^2}\int_0^\infty dx\, e^{-\alpha x} =$$

$$= \frac{1}{2}\frac{\partial^2}{\partial\alpha^2}\frac{1}{\alpha} = \frac{1}{\cancel{2}}\cdot\cancel{2}\frac{1}{\alpha^3} = v_{th}^6$$

Rate of loss of KE is

$$J\cdot a = an\sqrt{\frac{2}{\pi m}}\,(k_BT)^{3/2}$$

↑ area of hole

$\boxed{\text{June 2006 - Q7}}$

Mean # of fermions with energy ε_s is

$$n_s = \frac{g_s}{e^{\beta\varepsilon_s - \alpha} + 1}$$

β and α are Lagrange multipliers that arose from maximising entropy at fixed mean energy and # of particles.

$\beta = \frac{1}{k_B T}$ temperature $\quad \alpha = \beta\mu$ chemical potential

g_s is the # of microstates that have energy ε_s (discrete analog of density of states $g(\varepsilon)\,d\varepsilon$)

At absolute 0 ($T=0$),

$$U = \frac{3}{5} N \varepsilon_F \quad , \quad \varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \qquad (\text{set } 2s+1 = 2)$$

[derived on p. 206]

U is extensive (electrons are non-interacting):
double the size of the system $N \to 2N$, $V \to 2N$, then $\varepsilon_F \to \varepsilon_F$ and $U \to 2U$

But Just doubling N ~~in~~ a fixed V increases the density of the system and, therefore, increases ε_F.
So $U \propto N^{5/3}$ increases with N faster than linear.

Calculation of μ for $\overset{\overset{N\text{ states}}{\downarrow}}{\varepsilon_s = 0}$ and $\overset{\overset{gN\text{ states}}{\swarrow}}{\varepsilon}$ being the only single-particle levels:

$$\cancel{N} = \frac{\cancel{N}}{e^{-\beta\mu}+1} + \frac{\cancel{N} g}{e^{\beta(\varepsilon-\mu)}+1}$$

$$1 = \frac{1}{e^{-\beta\mu}+1} + \frac{g}{e^{\beta(\varepsilon-\mu)}+1} \approx 1 - e^{-\beta\mu} + g\, e^{-\beta(\varepsilon-\mu)}$$

because $\beta\mu \sim \dfrac{\varepsilon_F}{k_BT} \gg 1$ for $k_BT \ll \varepsilon_F$

and $\beta(\varepsilon-\mu) \sim \dfrac{\varepsilon_F}{k_BT}$ as well.

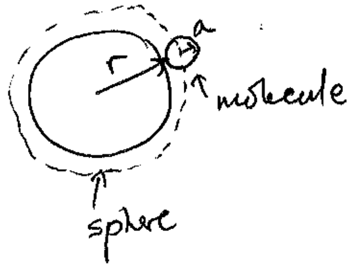
$$-\beta\mu = -\beta(\varepsilon-\mu) + \ln g$$

$$\mu = \frac{1}{2}\left(\varepsilon - k_BT \ln g\right) \quad \text{q.e.d.}$$

(Note that this means $\varepsilon_F = \frac{1}{2}\varepsilon$ and so our a priori estimates were indeed correct)

June 2005 - Q7

1)



Flux of molecules onto the sphere

$$\Phi = \frac{1}{4} n \langle v \rangle \quad \text{(p. 202)}$$

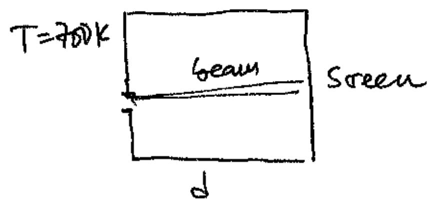
In order to collide with the sphere, the molecules must cross the sphere of radius $r + a$

r: sphere's radius; a: molecular radius

So, rate of collisions with sphere is

$$\frac{1}{4} n \langle v \rangle \cdot 4\pi (r+a)^2 = \pi n \langle v \rangle (r+a)^2$$

2)



Effusing particle distribution per unit time:

$$d\Phi \propto v^3 e^{-v^2/v_{th}^2}$$

So, mean speed of particles that hit the screen per unit time:

$$v_1 = \frac{\int_0^\infty dv\, v^4 e^{-v^2/v_{th}^2}}{\int_0^\infty dv\, v^3 e^{-v^2/v_{th}^2}} = \frac{3}{4}\sqrt{\pi}\, v_{th}$$

(Gaussian integrals done in the usual way)

~~[illegible]~~ # of particles (with velocity $\vec{v}$) that emerged over time t is $\propto d\Phi \cdot t$. They spread over distance vt.

So # of them over distance d between hole & screen:

$$\propto d\Phi \cdot t \cdot \frac{d}{vt} = \frac{d\Phi}{v} \propto v^2 e^{-v^2/v_{th}^2}$$

(back to Maxwellian distribution)

So, mean speed at an instant between hole and screen:

$$v_2 = \frac{\int_0^\infty dv\, v^3 e^{-v^2/v_{th}^2}}{\int_0^\infty dv\, v^2 e^{-v^2/v_{th}^2}} = \sqrt{\frac{4}{\pi}}\, v_{th} \quad , \; v_{th} = \sqrt{\frac{2k_BT}{m}}$$

3) Faster atoms in the beam simply "see" a gas of nearly immobile atoms in the chamber, with density $n = \frac{p}{k_BT}$.

Cross-section: $\sigma = \pi(2a)^2 = 4\pi a^2$

Fast atom ~~cccc~~ covers volume σL if it travels distance L. $L = \lambda_{mfp}$ if there is one ambient atom in this volume:

$$\sigma \lambda_{mfp} \cdot n = 1 \quad \Rightarrow \quad \lambda_{mfp} = \frac{1}{\sigma n} = \frac{k_BT}{4\pi a^2 p}$$

$(\approx 6.7\,\mathrm{m})$

Atoms at 10 m/s are slow compared to the ambient atoms.

So the situation is like that for a stationary sphere sitting in a gas, waiting to be collided with.

We calculated the rate of collisions to be

$$\pi n \langle v\rangle (\underset{a}{\underset{\parallel}{r}}+a)^2 = 4\pi a^2 n \underset{\uparrow}{\langle v\rangle}$$

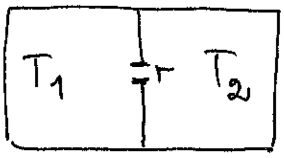
mean speed in chamber $= \sqrt{\frac{8k_BT}{\pi m}}$

Mean free path then is the distance this sphere travels (slowly!) until the first collision:

$$\lambda_{mfp} = \frac{v^{\leftarrow 10\,\mathrm{m/s}}}{4\pi a^2 n \langle v\rangle} = \frac{1}{\sigma n}\frac{v}{\langle v\rangle} \quad (\approx 0.13\,\mathrm{m})$$

$\boxed{\text{June 2003 – Q8}}$

• Derivation of $\Phi = \frac{1}{4} n \langle v \rangle$ – see p. 202

• 

(a) $r \gg \lambda_{mfp}$ in both chambers. Equilibrium is achieved when pressures equalize:

$$p_1 = p_2$$

(otherwise there will be a flow from the chamber with higher pressure to that with lower one)

$$n_1 k_B T_1 = n_2 k_B T_2 \quad \Rightarrow \quad \frac{n_1}{n_2} = \frac{T_2}{T_1}$$

$$\frac{\lambda_1}{\lambda_2} = \frac{n_2}{n_1} = \frac{T_1}{T_2} = \frac{1}{4}$$

(b) $r \ll \lambda_{mfp}$ in both chambers – effusion regime. Equilibrium when particle fluxes equalise:

$$\Phi_1 = \Phi_2$$

$$\frac{1}{4} n_1 \langle v_1 \rangle = \frac{1}{4} n_2 \langle v_2 \rangle \quad \Rightarrow \quad \frac{n_1}{n_2} = \frac{\langle v_2 \rangle}{\langle v_1 \rangle} = \sqrt{\frac{T_2}{T_1}}$$

because $\langle v \rangle \propto v_{th} \propto \sqrt{T}$

So $$\frac{\lambda_1}{\lambda_2} = \sqrt{\frac{T_1}{T_2}} = \frac{1}{2}$$