Statistical Physics (A1) — A. Schekochihin
Oxford HT-2012

§§ 14-16. Quantum Gases — SUMMARY

$$\ln \mathcal{Z} = \pm \sum_{i \leftarrow \text{single-particle microstates}} \ln\left[1 \pm e^{-\beta(\varepsilon_i - \mu)}\right]$$

single-particle energies (ε_i)

grand partition function

⊕ fermions, $n_i = 0, 1$ Pauli exclusion principle. Spin $\frac{1}{2}$-integer

⊖ bosons, $n_i = 0, 1, 2 \ldots$ Spin integer.

Occupation number statistics:

$$\boxed{\bar{n}_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)} \pm 1}}$$

⊕ Fermi-Dirac

⊖ Bose-Einstein

Construction of thermodynamics:

NB: 3D, non-relativistic $\varepsilon(\vec{k}) = \frac{\hbar^2 k^2}{2m}$

1) Find $\mu = \mu(n, T)$ from

$$\bar{N} = \sum_i \bar{n}_i = \frac{(2S+1)Vm^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\infty \frac{d\varepsilon \sqrt{\varepsilon}}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

2) Energy

$$U = \sum_i \varepsilon_i \bar{n}_i = \frac{(2S+1)Vm^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\infty \frac{d\varepsilon\, \varepsilon^{3/2}}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

grand potential: $\Phi = -PV = -\frac{2}{3} U$

$\Rightarrow$ equation of state

3) Entropy $S = \dfrac{U - \Phi - \mu \bar{N}}{T}$ $\Rightarrow$ heat capacities.

4) Adiabatic process: $PV^{5/3} = \text{const}$ if $S, \bar{N} = \text{const}$.

Classical Limit:

$$e^{\beta\mu} \approx \frac{n\lambda_{th}^3}{2S+1} \ll 1 \quad , \quad \lambda_{th} = \hbar\sqrt{\frac{2\pi}{m k_B T}}$$

$$\bar{n}_i \approx \frac{n\lambda_{th}^3}{2S+1} e^{-\beta\varepsilon_i} \Rightarrow F(\vec{v})\, d^3\vec{v} = \frac{n}{(2\pi k_B T/m)^{3/2}} e^{-\frac{mv^2}{2k_B T}}$$

Maxwell-Boltzmann distribution

$n\lambda_{th}^3 \ll 1 \Leftrightarrow$ # of quantum states of a single particle $\gg$ # of particles

$\Leftrightarrow$ particle position uncertainty $\ll$ mean interparticle distance

Satisfied in hot, dilute gases.

Fails, e.g., for electrons in metals, white dwarves, neutron stars

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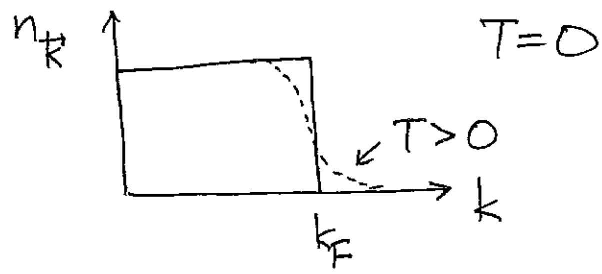
Degenerate Fermi Gas:



States occupied up to

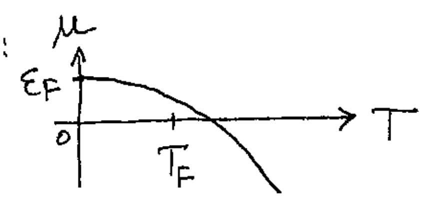
$$k_F = \left(\frac{6\pi^2 n}{2S+1}\right)^{1/3}$$

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \mu(T=0) = k_B T_F \text{ Fermi temperature}$$

Fermi energy

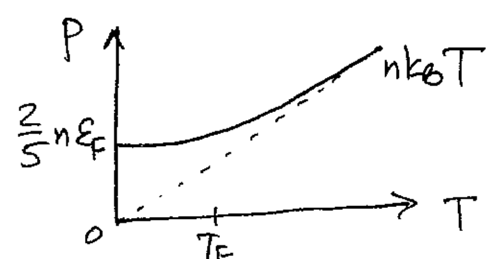
approximation valid: $n\lambda_{th}^3 \gg 1 \Leftrightarrow T \ll T_F$

With finite-temperature corrections:



$$\mu = \varepsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \dots\right]$$

$$U = \frac{3}{5}N\varepsilon_F\left[1 + \frac{5\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \dots\right]$$

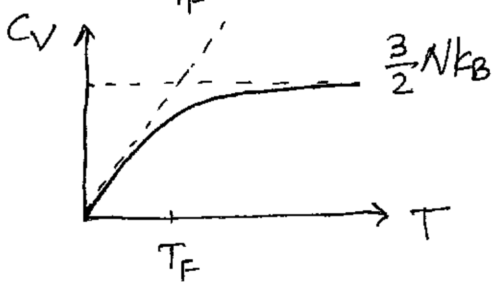


"pure mechanism" at T=0,

$$p = \frac{2}{5}n\varepsilon_F\left[1 + \frac{5\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \dots\right]$$

$$C_V = Nk_B\frac{\pi^2}{2}\frac{k_B T}{\varepsilon_F} + \dots$$

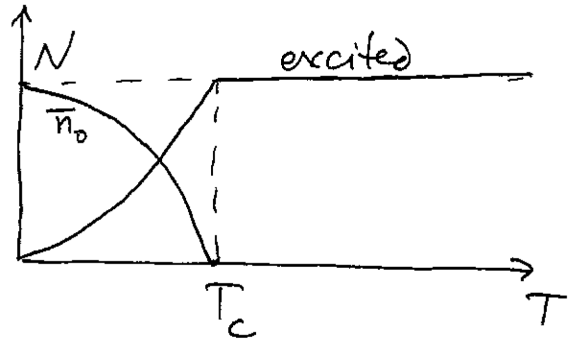
(dominant over lattice contributions for electrons in metals at low T)



Degenerate Bose gas:

Zero-energy condensate forms

for $T < T_c = \frac{2\pi\hbar^2}{m k_B}\left[\frac{n}{2.612(2S+1)}\right]^{2/3}$

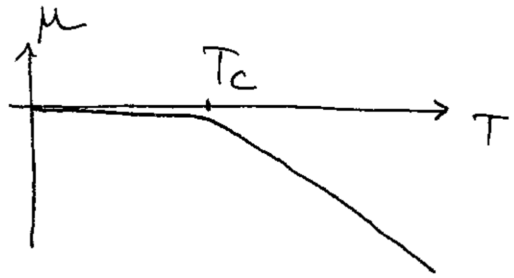


Number of particles with $\varepsilon = 0$:

$$\frac{\bar{n}_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

(Bose-Einstein Condensation).

Chemical potential

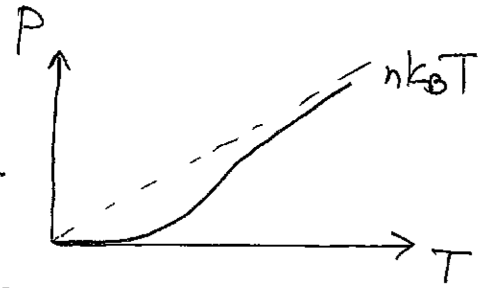


$$\mu(T<T_c) \sim -\frac{k_B T}{N} \to 0-$$

Energy (ε≠0 particles not conserved)

$$U(T<T_c) \approx 0.77 N k_B T_c \left(\frac{T}{T_c}\right)^{5/2}$$

Equation of state

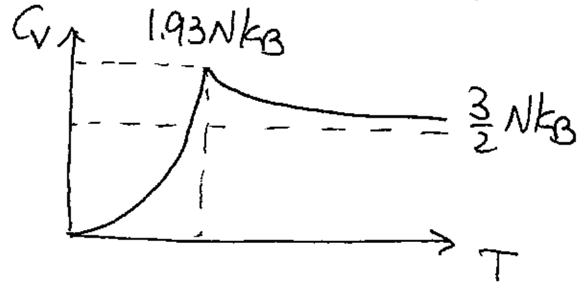


indep. of V

$$p(T<T_c) \approx 0.085\,(2S+1)\,\frac{m^{3/2}(k_B T)^{5/2}}{\hbar^3}$$

Heat capacity

$$C_V(T<T_c) \approx 1.93 N k_B \left(\frac{T}{T_c}\right)^{3/2}$$



Photon gas: bosons, $\mu = 0$, $\varepsilon = \hbar k c$

— coming up —