§5. Derivation of Fluid Equations with Collisional Transport.

5.1 Kinetic Equation

Recall that in an inhomogeneous system, the particle distribution must be characterised by a pdf that dep's both on $\vec{v}$ and $\vec{r}$: $F(\vec{r},\vec{v})$

and we agreed to normalise it so

$$\int d^3\vec{r}\int d^3\vec{v}\, F(\vec{r},\vec{v}) = N \quad \#\text{ of particles}$$

also $\int d^3\vec{v}\, F(\vec{r},\vec{v}) = n(\vec{r})$ density

[in a homogeneous system, $F = F(\vec{v})$ indep. of $\vec{r}$ and the pdf we used for prev. calculations was $f(\vec{v}) = \frac{F(\vec{v})}{n}$]

If we knew $F(\vec{r},\vec{v})$, we could also calculate all the other quantities of interest:

Mean flows: $\int d^3\vec{v}\,\vec{v}\,F(\vec{r},\vec{v}) = n(\vec{r})\,\vec{u}(\vec{r})$ particle flux

Temperature: this must come from the mean energy:

$$\int d^3\vec{v}\,\frac{mv^2}{2}F(\vec{r},\vec{v}) \overset{\uparrow}{=} \int d^3\vec{w}\,\frac{m(\vec{u}+\vec{w})^2}{2}F(\vec{r},\vec{v}) =$$

$$\vec{v} = \underbrace{\vec{u}}_{\text{ordered motion}} + \underbrace{\vec{w}}_{\text{disordered motion (heat!)}}$$ "peculiar velocity"

$$\int d^3\vec{w}\,\vec{w}\,F(\vec{r},\vec{v}) = 0$$

$$= \frac{mnu^2}{2} + n\left\langle\frac{mw^2}{2}\right\rangle \equiv \varepsilon \qquad \varepsilon = \frac{3}{2}nk_BT$$

↑ avg. kinetic energy of disordered motion

$$= \frac{mnu^2}{2} + \varepsilon$$

[recall $\frac{1}{2}k_BT$ = (disordered) energy per particle per deg. of freedom]

Thus, we need an equation for $F(\vec{r},\vec{v})$ — from which we can then derive equations for $n, \vec{u}, T$.

Simplest derivation: pdf at time $t+\Delta t$

$$F(t+\Delta t,\vec{r},\vec{v}) = F(t,\vec{r}-\vec{v}\Delta t,\vec{v}) + \Delta F_{coll}$$

particles found at $\vec{r}$ w/ velocity $\vec{v}$ at time $t+\Delta t$ were at $\vec{r}-\vec{v}\Delta t$ at time t

whatever correction must be made to F to account for collisions during Δt

$$\approx F(t,\vec{r},\vec{v}) - \vec{v}\cdot\nabla F\,\Delta t + \Delta F_{coll}$$

$$\boxed{\frac{\partial F}{\partial t} + \vec{v}\cdot\nabla F = \left(\frac{\partial F}{\partial t}\right)_{coll}}$$

NB: we assume no forces act on particles besides collisions
Ex. What if forces act on particles?

"collision operator" was derived by Boltzmann

Kinetic equation

(conservation of density in phase space)

I will not present the derivation of the Boltzmann operator [see Kardar's book or LL-Vol.10].

Instead, I will use some simple criteria that any coll. operator must satisfy to produce a very simple model form for it.

1) If collisions are elastic, they should not change the total momentum & energy of particles - and should not also change the # of particles (no sticking!)

Therefore,

$$\int d^3\vec{v}\left(\frac{\partial F}{\partial t}\right)_{coll} \overset{\text{particle\#}}{=} 0, \quad \int d^3\vec{v}\, m\vec{v}\left(\frac{\partial F}{\partial t}\right)_{coll} = 0^{\text{momentum}}$$

$$\text{and} \quad \int d^3\vec{v}\, \frac{mv^2}{2}\left(\frac{\partial F}{\partial t}\right)_{coll} = 0^{\text{energy}}.$$

2) The argument I used to derive the Maxwellian distribution can be generalised to apply to a parcel of gas at some location $\vec{r}$, moving with mean velocity $\vec{u}(\vec{r})$ & having density $n(\vec{r})$ and temperature $T(\vec{r})$ – except the Max. distribution is now for peculiar velocities $\vec{w} = \vec{v} - \vec{u}(\vec{r})$. So we expect collisions to drive the distribution to a local Maxwellian – & so if it is a local Maxwellian, the coll. operator should not change it any further:

$$\left(\frac{\partial F_M}{\partial t}\right)_{coll} = 0 \quad \text{where} \quad F_M = \frac{n(\vec{r})\, m^{3/2}}{(2\pi k_B T(\vec{r}))^{3/2}}\, e^{-\frac{m|\vec{v}-\vec{u}(\vec{r})|^2}{2k_B T(\vec{r})}}$$

[for Boltzmann operator, it can be proved that

$$\left(\frac{\partial F}{\partial t}\right)_{coll} = 0 \quad \text{implies} \quad F = F_M$$

& also that collisions will always be attempting to relax the distribution $F \to F_M$ – this is associated with the so called Boltzmann's H-theorem, which is the law of entropy increase for kinetic systems – look it up if int'd; see Kardar or LL-Vol 10]

3) The time scale on which this relaxation to a Maxwellian happens is roughly the collision rate $\nu_c \sim \sigma n v_{th}$ [depends on n, σ, T, so can be function of $\vec{r}$; in reality can also be function of $\vec{v}$ [illegible]]

or Bhatnagar-Gross-Krook
BGK

So, let's postulate

$$\left(\frac{\partial F}{\partial t}\right)_{coll} = -\nu_c (F - F_M) \equiv -\nu_c \delta F$$ "Krook operator"

Note that conservation of particles, momentum and energy means that

$$\int d^3\vec{v}\, \delta F = 0 \quad , \quad \int d^3_{\vec{v}}\, m\vec{v}\, \delta F = 0 \quad \text{and} \quad \int d^3\vec{v}\, \frac{mv^2}{2}\, \delta F = 0$$

i.e. we declare, by definition that

$$F = F_M + \delta F$$

where F_M contains all of $n, \vec{u}, T$.

NB: The Krook operator is way too simplified of course but it will serve adequately for our purpose of illustrating how transport equations are derived

[it is also instructive to see how this kind of modelling is done in physics — it is not always possible to calculate everything exactly and so we often have to invent simplified models — a skill also useful in other applications of rational intellect to life]

5.2. Moment equations

We now take moments of the kinetic equation:

Density

$$\frac{\partial}{\partial t} n = \frac{\partial}{\partial t}\int d^3\vec v\, F(\vec r,\vec v) = \int d^3\vec v\, \frac{\partial F}{\partial t} =$$

$$= \int d^3\vec v\left[-\vec v\cdot\nabla F + \underbrace{\left(\frac{\partial F}{\partial t}\right)_{coll}}_{0\text{ by particle conservation}}\right] =$$

$$= -\nabla\cdot\underbrace{\int d^3\vec v\,\vec v\, F}_{\text{flux of particles}} = -\nabla\cdot(n\vec u)$$

$$\boxed{\frac{\partial n}{\partial t} + \nabla\cdot(n\vec u) = 0}$$

Continuity equation

$\frac{\partial N}{\partial t} = -\oint d\vec S\cdot(\vec u n)$ flux through boundaries

NB: Particle # conserved $\frac{\partial N}{\partial t} = 0$ (integrate over closed box)

NB: NOT closed! need to know $\vec u$!

Note: $$\underbrace{\frac{\partial n}{\partial t} + \overset{\text{density carried by fluid flow}}{\vec u\cdot\nabla n}}_{\text{conv. derivative}} + \underset{\uparrow\text{ compression/rarefication}}{n\nabla\cdot\vec u} = 0$$

Incompressible medium: $n = \text{const} \Rightarrow \boxed{\nabla\cdot\vec u = 0}$

(it turns out that this is a good approximation when gas moves subsonically: $u \ll c_s \sim v_{th}$ speed of sound) $\leftarrow M = \frac{u}{c_s} \ll 1$ Mach #

Momentum:

$$\int d^3\vec v\, m\vec v\,\frac{\partial F}{\partial t} = \frac{\partial}{\partial t}\underset{\text{density of momentum}}{mn\vec u} = mn\frac{\partial\vec u}{\partial t} + \underbrace{m\vec u\frac{\partial n}{\partial t}}_{\substack{\| \\ -m\vec u\,\nabla\cdot(n\vec u)}}$$

$-m\vec u\,\nabla\cdot(n\vec u)$ ⊛ cancels with ⊛⊛ (p.41)

from kinetic equation

$$= \int d^3\vec v\left[-m\vec v\,\vec v\cdot\nabla F + m\vec v\underbrace{\left(\frac{\partial F}{\partial t}\right)_{coll}}_{\text{by momentum conservation}}\right] =$$

[redacted] $\vec v = \vec u + \vec w$

$$= -\nabla\cdot\underbrace{\int d^3\vec v\, m\vec v\vec v F}_{\text{flux of momentum}} = -\nabla\cdot\int d^3\vec v\, m(\vec u+\vec w)(\vec u+\vec w)F$$

$$= -\nabla\cdot(mn\vec u\vec u) - \nabla\cdot\underbrace{\int d^3\vec v\,\vec w\vec w F}_{\equiv\hat P\text{ pressure tensor}} =$$

$$= -mn\,\vec{u}\cdot\nabla\vec{u} - \underbrace{\cancel{\left(m\vec{u}\,\nabla\cdot(n\vec{u})\right)}}_{(**)} - \nabla\cdot\hat{\underline{P}}$$

cancels with (*) (p.40)

L.5 ended here.

Pressure tensor can be split into <u>pressure</u> + <u>stress tensor</u>:

$$\hat{\underline{P}} = \int d^3\vec{v}\,m\vec{w}\vec{w}\,F = \underbrace{\int d^3\vec{v}\,m\vec{w}\vec{w}\,F_M}_{\parallel\ \text{because } F_M \text{ is iso.}} + \underbrace{\int d^3\vec{v}\,m\vec{w}\vec{w}\,\delta F}_{\equiv\ \hat{\Pi}}$$

$\langle\vec{w}\vec{w}\rangle_{\text{angle}} = \mathbb{1}\frac{1}{3}\langle w^2\rangle$

or $\langle w_i w_j\rangle_{\text{angle}} = \frac{1}{3}\langle w^2\rangle\delta_{ij}$

$$\underbrace{\mathbb{1}\,\frac{1}{3}\int d^3\vec{v}\,w^2 F_M}_{\equiv}$$

$p = nk_BT$ standard pressure

T is the temp. of the Maxwellian

So, finally, ← conv. derivative

$$\boxed{mn\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p - \nabla\cdot\hat{\Pi}}\,, \quad \hat{\Pi} = \int d^3\vec{w}\,m\vec{w}\vec{w}\,\delta F$$

can change int'n to $\vec{w}$

velocity changing at a spot

velocity carried by flow ("advection")

pressure force

stress. <u>Viscosity</u> will come from here, i.e. from the non-Max. part of the distr. function

<u>Momentum equation</u>.

NB: NOT closed! Need to know p and $\hat{\Pi}$

Ⓠ If we now calculate 2nd moment (energy), will we be able to close the equation for it?

A pattern emerges: we ~~calla~~ integrate the kinetic eqn to get evolution of moments ($1\to n$, $\vec{v}\to\vec{u}$, $v^2\to\varepsilon$ or p) but they turn out to depend on next-order moments.

This is because of the $\vec{v}\cdot\nabla F$ term.

So will eventually need to solve for δF to calculate anyth.

Energy.

$$\int d^3\vec{v}\,\frac{mv^2}{2}\frac{\partial F}{\partial t} = \underbrace{\frac{\partial}{\partial t}\frac{mnu^2}{2}} + \frac{\partial}{\partial t}\overbrace{n\left\langle\frac{mw^2}{2}\right\rangle}^{\mathcal{E}\text{ energy density}}$$

$$\mathcal{E} = \frac{3}{2}nk_BT = \frac{3}{2}p$$

(NB: the integral above has no δF because $\int d^3\vec{v}\,\frac{mv^2}{2}\delta F = 0$!)

$$\cancel{\frac{mu^2}{2}\frac{\partial n}{\partial t} + mn\vec{u}\cdot\frac{\partial \vec{u}}{\partial t}}$$

$$\cancel{= -\frac{mu^2}{2}\nabla\cdot(n\vec{u}) - mn(\vec{u}\cdot\nabla\vec{u})\cdot\vec{u}} - (\nabla\cdot\hat{P})\cdot\vec{u}$$

(*) cancels with (**) below

from kinetic equation

$$= \int d^3\vec{v}\left[-\frac{mv^2}{2}\overset{(\vec{u}+\vec{w})}{\vec{v}}\cdot\nabla F + \frac{mv^2}{2}\cancel{\left(\frac{\partial F}{\partial t}\right)_{coll}}\right] =$$

0 by energy conservation

Ex.

$$= -\nabla\cdot\int d^3\vec{v}\,\frac{mv^2}{2}\vec{v}F =$$

flux of kinetic energy

$$= -\nabla\cdot\left[\vec{u}\underbrace{\int d^3\vec{v}\,\frac{mv^2}{2}F}_{=\frac{mnu^2}{2}+\mathcal{E}}\ \text{[redacted]} + \text{[redacted]}\int d^3\vec{v}\,\vec{w}\left(\cancel{\frac{mu^2}{2}} + \frac{mw^2}{2} + m\vec{u}\cdot\vec{w}\right)F\right]$$

0 because $\int d^3\vec{v}\,\vec{w}F = 0$

$$= -\nabla\cdot\left[\frac{mnu^2}{2}\vec{u} + \mathcal{E}\vec{u} + \underbrace{\int d^3\vec{w}\,\frac{mw^2}{2}\vec{w}F}_{=\vec{J}} + \vec{u}\cdot\underbrace{\int d^3\vec{w}\,m\vec{w}\vec{w}F}_{=\hat{P}}\right]$$

flow of kinetic energy

flow of thermal energy

$\vec{J}$ heat flux

$$= \cancel{-\frac{mu^2}{2}\nabla\cdot(n\vec{u}) - mn\vec{u}\cdot\nabla\frac{u^2}{2} - (\nabla\cdot\hat{P})\cdot\vec{u}} - \hat{P}:\nabla\vec{u} - \nabla\cdot(\mathcal{E}\vec{u}) - \nabla\cdot\vec{J}$$

↑ means $\hat{P}_{ij}\partial_i u_j$

(**) cancels with (*) above

We have got the energy equation

$$\boxed{\underbrace{\frac{\partial \mathcal{E}}{\partial t} + \nabla\cdot(\mathcal{E}\vec{u})} = -\nabla\cdot\vec{J} - \hat{P}:\nabla\vec{u}}\quad , \quad \vec{J} = \int d^3\vec{w}\,\frac{mw^2}{2}\vec{w}F$$

heat flux

$$= \int d^3\vec{w}\,\frac{mw^2}{2}\vec{w}\,\delta F$$

(no heat flux in the Maxwellian! – thermal equilibrium!)

Thermal conductivity will come from here.

this is like cont. eqn but for energy density $\mathcal{E} = \frac{3}{2}nk_BT$

heating

$$p\nabla\cdot\vec{u} + \hat{\Pi}:\nabla\vec{u}$$

compressional viscous

conversion of the energy of flows into disordered particle motion

We know it is heating because it is an "inhom" term in the $\mathcal{E}$ eqn, not in flux form

NB: As expected, again NOT closed eqn – need to know $\vec{J}$

[Note that integrating over some volume,

$$\frac{\partial}{\partial t}\int_V d^3\vec{r}\,\mathcal{E} = -\int_{\partial V} d\vec{S}\cdot\vec{J} - \int_{\partial V} d\vec{S}\cdot(\vec{u}\mathcal{E}) - \int_V d^3\vec{r}\,\hat{P}:\nabla\vec{u}\ ]$$

heat flow through boundary

energy flow with fluid through boundary (if permeable)

volume-distr. heating.

Finally, can convert into temperature equation:

$$\frac{\partial\mathcal{E}}{\partial t} + \nabla\cdot(\mathcal{E}\vec{u}) = \frac{3}{2}nk_B\frac{\partial T}{\partial t} + \underbrace{\cancel{\frac{3}{2}k_BT\frac{\partial n}{\partial t}} + \cancel{\frac{3}{2}k_BT\nabla\cdot(n\vec{u})}}_{\text{cont. equation}}$$

$$+ \frac{3}{2}k_Bn\vec{u}\cdot\nabla T$$

$$\boxed{\frac{3}{2}nk_B\left(\frac{\partial T}{\partial t} + \vec{u}\cdot\nabla T\right) = -\underset{\uparrow}{\nabla\cdot\vec{J}} - \underset{\uparrow}{nk_BT\nabla\cdot\vec{u}} - \underset{\uparrow}{\hat{\Pi}:\nabla\vec{u}}}$$

"compr. heating"

need to calculate these, so must have δF!

Ex. Work out and interpret what happens to total energy $\frac{mnu^2}{2} + \mathcal{E}$

5.3 Collisional Transport

Note that so far we have not used the _explicit form_ of the collision operator — only its conservation properties. Now we _will_ use it because we want to calculate δF.

$$\frac{\partial F}{\partial t} + \vec{v}\cdot\nabla F = -\nu_c (F - F_M) \equiv -\nu_c \delta F$$

$$\left.\begin{array}{ccc} \sim & \sim & \sim \\ \omega F & \frac{v}{\ell} F & \nu_c F \\ \uparrow & \uparrow & \uparrow \\ \text{rate of change of } F & \text{scale of var. of } F & \text{coll. rate.} \end{array}\right\} \quad \text{If } \nu_c \gg \omega, \frac{v}{\ell} \text{ then } \delta F \ll F_M,$$

pdf is close to Maxwellian

$$\boxed{\delta F = -\frac{1}{\nu_c}\left(\frac{\partial F_M}{\partial t} + \vec{v}\cdot\nabla F_M\right)} \ll F_M$$

provided F_M (i.e., $n, \vec{u}, T$) change slowly in time and space compared to collisions.

Recall that all we needed was to calculate

$$\hat{\Pi} = \int d^3\vec{v}\, m\vec{w}\vec{w}\,\delta F \quad \text{and} \quad \vec{J} = \int d^3\vec{v}\, \frac{mw^2}{2}\vec{w}\,\delta F$$

We now know δF in terms of F_M, which we know how to express in terms of $n, \vec{u}, T$:

$$F_M = \frac{m^{3/2} n}{(2\pi k_B T)^{3/2}} e^{-\frac{m|\vec{v}-\vec{u}|^2}{2k_B T}}$$

So, if we calculate those integrals, we'll have $\hat{\Pi}$ and $\vec{J}$ in terms of $n, \vec{u}, T$ and our equations are closed!

NB: We needed a simplified coll. operator because otherwise would have has to invert $(\partial F/\partial t)_{coll}$ — hard (see LL-10, Kardar)

Let us first do the simplest case – <u>Static</u>, $\boxed{\vec{u} = 0}$

Then, from conti. eqn,

$$\frac{\partial n}{\partial t} = 0, \quad n = \text{const in time}$$

From momentum equation,

$$\nabla p = 0 \quad \text{pressure balance} \quad \left(\Rightarrow \frac{\nabla n}{n} = \frac{\nabla T}{T} \right)$$

From temperature equation,

$$\frac{3}{2} n k_B \frac{\partial T}{\partial t} = -\nabla \cdot \vec{J}$$

$$\vec{J} = \int d^3\vec{w}\, \frac{m w^2}{2} \vec{w} \left[-\frac{1}{\nu_c} \left(\frac{\partial F_M}{\partial t} + \vec{v} \cdot \nabla F_M \right) \right] =$$

$\vec{w} = \vec{v}$ because $\vec{u} = 0$

integral vanishes because no heat flux in F_M

$\langle \vec{v}\vec{v} \rangle_{\text{angle}} = \frac{1}{3} \mathbb{1} v^2$

$$= -\frac{1}{\nu_c} \int d^3\vec{v}\, \frac{m v^2}{2} \vec{v}\, \vec{v} \cdot \nabla F_M = -\frac{m}{2\nu_c} \nabla \cdot \int d^3\vec{v}\, \vec{v}\vec{v}\, v^2 F_M =$$

$$= -\frac{m}{6\nu_c} \nabla \cdot \left(n \langle v^4 \rangle \right) = -\frac{5}{2} \frac{k_B}{\nu_c m} \nabla \left(\underbrace{n k_B}_{p} T^2 \right) =$$

$\langle v^4 \rangle$: $\frac{15}{4} v_{th}^4 = 15 \left(\frac{k_B T}{m} \right)^2$

$n k_B T^2 = p T$, but $p = \text{const}$

$$= -\frac{5}{2} \frac{k_B}{\nu_c m} p \nabla T = -\boxed{\frac{5}{4} n k_B \frac{v_{th}^2}{\nu_c}} \nabla T$$

$\parallel$ $\varkappa$ <u>thermal conductivity</u>

(prefactor $\frac{5}{4}$ cannot be trusted because we used simplified coll. operator)

$$\frac{3}{2} n k_B \frac{\partial T}{\partial t} = \nabla \cdot \left(\frac{5}{2} \frac{k_B}{\nu_c m} p \nabla T \right)$$

$$= \varkappa \nabla^2 T$$

Do I really have the right to pull this from under ∇? See next page

Note that technically speaking ν_c must depend on temperature: in fact, we know that

$$\nu_c \sim \sigma n v_{th} \propto n\sqrt{T}$$

The way to argue for assuming it constant is

$$n = n_0 + \delta n \, , \quad T = T_0 + \delta T$$

where n_0 and T_0 = const (i.e. the departures from global qq. are small)

Note also that $\varkappa \sim \dfrac{k_B n k_B T}{m \sigma n \sqrt{T/m}} \sim k_B^2 \sqrt{\dfrac{T}{m}}$

independent of n

thermal diffusivity

Finally, $\boxed{\frac{\partial T}{\partial t} = D\nabla^2 T}$ thermal diffusion equation

q.e.d!

$$D = \frac{\varkappa}{\frac{3}{2}nk_B} = \frac{5}{3}\frac{p}{mn\nu_c} = \frac{5}{3}\frac{k_B T}{m\nu_c} = \frac{5}{6}\frac{v_{th}^2}{\nu_c}$$

Note that D depends on T, of course, so technically speaking the equation is nonlinear, but in ~~[illegible]~~ many of our practical situations, $T = \underset{\substack{\shortparallel\\ \text{const}}}{T_0} + \delta T$, $\delta T \ll T_0$

and so we can neglect this effect.

(but not in various violent explosive situations, for ex.)

Let us now restore flows: $\delta F = -\frac{1}{\nu_c}\left(\frac{\partial F_M}{\partial t} + \underset{\substack{\uparrow\\ \vec{u}+\vec{w}}}{\vec{v}}\cdot\nabla F_M\right)$

~~[illegible]~~

We won't attempt the full calculation (it is quite tedious - you can try it), but just consider the simpler case where the flows are relatively slow, viz. subsonic:

$u \ll c_s \sim v_{th}$, so $w \gg u$ (disordered motion much faster than ordered) and $\vec{v} \approx \vec{w}$

~~[illegible]~~ Then

$$F_M = \frac{n}{(\pi v_{th}^2)^{3/2}} e^{-\frac{|\vec{v}-\vec{u}|^2}{v_{th}^2}} \approx \underbrace{\frac{n\, e^{-v^2/v_{th}^2}}{(\pi v_{th}^2)^{3/2}}}_{\substack{|||\\ F_M^{(0)}}} \left[1 + \frac{2\vec{v}\cdot\vec{u}}{v_{th}^2} + \dots\right]$$

- In the calculation of heat flux, we can simply ignore all terms that contain $\vec{u}$ (because $\frac{u}{v_{th}}$ is small) and, to lowest order, recover the same result.
- In the calculation of stress tensor, we obviously need to keep $\vec{u}$ as stress will end up depend upon it.

To get the simplest version of this calculation, let us neglect all variation of n and T (so $F_M^{(0)}$ = const in space and time). Then

$$\hat{\Pi} = \int d^3\vec{v}\, m\vec{w}\vec{w}\left[-\frac{1}{\nu_c}\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)F_M^{(0)}\left(1+\frac{2\vec{v}\cdot\vec{u}}{v_{th}^2}\right)\right]$$

$\vec{w}\approx\vec{v}$

this disappears because motions are slow $\frac{\partial}{\partial t}\sim\frac{u}{\ell}\ll\frac{v}{\ell}$

this disappears because $F_M^{(0)}$ = const

NB: we already have one power of $\vec{u}$, so can neglect it everywhere else!

$$\approx -\frac{1}{\nu_c}\int d^3\vec{v}\, m\vec{v}\vec{v}\,\frac{2\vec{v}\cdot(\nabla\vec{u})\cdot\vec{v}}{v_{th}^2}F_M =$$

$$= -\frac{2m}{\nu_c v_{th}^2}\left[\int d^3\vec{v}\, v_i v_j v_k v_l F_M\right]\partial_k u_l$$

$\langle v_i v_j v_k v_l\rangle_{angle} = \frac{v^4}{15}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$ Ex.

$$= -\frac{2}{15}\frac{mn}{\nu_c v_{th}^2}\underbrace{\langle v^4\rangle}_{=\frac{15}{4}v_{th}^4}\left[\nabla\vec{u}+(\nabla\vec{u})^T\right]$$

NB: $\nabla\cdot\vec{u}=0$ because n = const

$$= -\frac{1}{2}\frac{mnv_{th}^2}{\nu_c}\left[\nabla\vec{u}+(\nabla\vec{u})^T\right]$$

$\equiv \eta$ viscosity

(again, don't trust coefficient!)

Finally, the momentum equation is

$$mn\left(\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}\right)=-\nabla p+\underbrace{\nabla\cdot\left[\eta\left(\nabla\vec{u}+(\nabla\vec{u})^T\right)\right]}_{\substack{\parallel \\ \eta\nabla^2\vec{u}}}$$

because $\nabla\cdot\vec{u}=0$ and $\eta=\text{const}$

$$\boxed{\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}=-\frac{\nabla p}{mn}+\nu\nabla^2\vec{u}}$$

Navier-Stokes Equation

$$\nu=\frac{\eta}{mn}=\frac{1}{2}\frac{v_{th}^2}{\nu_c}$$

kinematic viscosity ("momentum diffusivity")

($1M Clay prize for provip slns exist!)

L6 ended here.

Note: You might wonder what ∇p is doing here if I said that ∇n and $\nabla T=0$. In fact there is a little bit of ∇p and it is determined by the condition $\nabla\cdot\vec{u}=0$:

$$\nabla\cdot\frac{\nabla p}{mn}=\nabla\cdot(\vec{u}\cdot\nabla\vec{u})$$

so $$\frac{\nabla p}{p}\sim\frac{mnu^2}{\ell p}\sim\frac{u^2}{v_{th}^2}\frac{1}{\ell}\ll\frac{1}{\ell}$$ (much smaller than scale of var. of u and $\sim M^2$ second order in Mach number)

In fact, it turns out that the temp. equation can peacefully coexist with the NS Equ:

$$\frac{3}{2}nk_B\left(\frac{\partial T}{\partial t}+\underset{\substack{\uparrow\\ \text{advection}}}{\vec{u}\cdot\nabla T}\right)=\underset{\substack{\uparrow\\ \text{diffusion}}}{\kappa\nabla^2T}+\eta|\nabla\vec{u}|^2$$

meaning $(\partial_i u_j)(\partial_i u_j)$ viscous heating