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BB Ch. 8
Pauli §26

§3. Collisions.

Treating gas as being in equilibrium involved assuming that molecules collide a suff. # of times for the initial conditions to be forgotten — this means there are certain constraints on the timescales on which we can believe the eq. distribution (how long do we wait for gas to Maxwellianise?) and on the spatial scales of the systems in which we can think of gas as being in collisional equilibrium:

$t \gg \tau_c$ "collision time" $\nu_c = \frac{1}{\tau_c}$ "coll. rate" (# of coll-s per unit time)

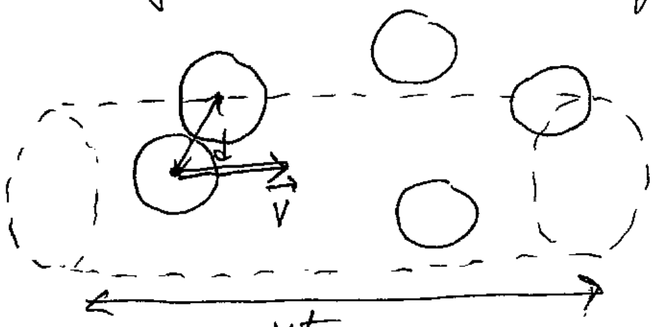
e.g. size of contain. → $l \gg \lambda_{mfp}$ "mean free path"

[recall that in the treatment of effusion, we needed a "small" hole, $d \ll \lambda_{mfp}$, to be able to snatch indiv. molecules w/o alerting the gas as a collective]

3.1. Cross-section

Assume particles are hard spheres of ~~radius~~ diameter d. So they collide if their centres approach within distance d of each other — so particle with velocity $\vec{v}$ in time t sweeps volume

$\sigma = \pi d^2$



vt

$\sigma v t$

↖ cross section

and will collide with anything in that volume.

3.2. Collision rate

Avg # of particles in volume $\sigma v t$ is

$$\sigma v t n$$

coll. time

$$t = \tau_c \text{ if } \sigma v \tau_c n = 1 \quad \text{(one collision)}$$

or, coll. rate is

$$\nu_c = \frac{1}{\tau_c} = \sigma n v$$

What is this velocity?

NB: can generalise σ to 1) particles of diff. diameters d_1, d_2 Ex.
2) "squashy" particles (not hard spheres) - then smaller σ at higher v
3) particles that act on each other with some force, e.g. Coulomb pot. for charged ("Rutherford x-section") → Ex. look it up!

Obviously, by order of magnitude,

$$v \sim \langle v \rangle \sim v_{rms} \sim v_{th}$$

So, a quick estimate is simply

$$\boxed{\tau_c \sim \frac{1}{\sigma n v_{th}} \sim \frac{1}{\sigma n}\sqrt{\frac{m}{T}}}$$

3.3 Mean free path

Can immediately define $\boxed{\lambda_{mfp} \sim v\tau_c \sim \frac{1}{\sigma n}} = \frac{k_B T}{\sigma p}$

independent of temperature!

So all we need to know about microphysics is
- size of molecules to get λ_{mfp}
- also their mass to get τ_c

3.4. Relative Speed

We can refine our definitions of τ_c and λ_{mfp} somewhat – although we do this just to understand the physics a bit better because fundamentally they are order-of-mag. quantities and it's not very meaningful to assign particularly precise values to them.

We had $\nu_c = \sigma n v$ ← velocity of particle travelling through gas of other particles.

But other particles are also moving, so actually it is <u>relative velocity</u>! $v = v_r$

So average over particle distribution:

$$\langle \nu_c \rangle = \sigma n \langle v_r \rangle$$

This involves velocities of 2 particles : $v_r = |\vec{v}_1 - \vec{v}_2|$

and so we need <u>joint</u> distribution $f(\vec{v}_1, \vec{v}_2)$:

$$\langle v_r \rangle = \int d^3\vec{v}_1 \int d^3\vec{v}_2 \, |\vec{v}_1 - \vec{v}_2| \, \underbrace{f(\vec{v}_1, \vec{v}_2)}_{= f(\vec{v}_1) f(\vec{v}_2)}$$

assume independent

$$= \iint d^3\vec{v}_1 \, d^3\vec{v}_2 \, |\vec{v}_1 - \vec{v}_2| \, \frac{1}{(\pi v_{th}^2)^3} \, e^{-v_1^2/v_{th}^2 - v_2^2/v_{th}^2}$$

Change variables: $(\vec{v}_1, \vec{v}_2) \to (\vec{V}, \vec{v}_r)$

$$\vec{V} = \frac{\vec{v}_1 + \vec{v}_2}{2}, \quad \vec{v}_r = \vec{v}_1 - \vec{v}_2$$

<u>Ex.</u> Show using this that $\langle v_r \rangle = \sqrt{2} \langle v \rangle$

↳ avg. speed of the Maxwellian distribution

Proof. $\quad v_1^2 + v_2^2 = 2V^2 + \frac{1}{2} v_r^2$

Jacobian $\quad \dfrac{\partial(\vec{v}_1, \vec{v}_2)}{\partial(\vec{V}, \vec{v}_r)} = 1$

So

$$f(\vec{v}_1, \vec{v}_2)\, d^3\vec{v}_1\, d^3\vec{v}_2 = d^3\vec{V}\, d^3\vec{v}_r \underbrace{\frac{e^{-\frac{2V^2 + \frac{1}{2}v_r^2}{v_{th}^2}}}{(\pi v_{th}^2)^3}}_{\equiv\, f(\vec{V}, \vec{v}_r)}$$

Then the distr. of relative velocities is

$$f(\vec{v}_r) = \int d^3\vec{V}\, f(\vec{V}, \vec{v}_r) = \text{const}\, e^{-\frac{v_r^2}{2v_{th}^2}}$$

$$\langle v_r \rangle = \int d^3\vec{v}_r\, v_r\, f(\vec{v}_r) = \frac{\int d^3\vec{v}_r\, v_r\, e^{-v_r^2/2v_{th}^2}}{\int d^3\vec{v}_r\, e^{-v_r^2/2v_{th}^2}}$$

$$\text{cf. } \langle v \rangle = \int d^3\vec{v}\, v\, f(\vec{v}) = \frac{\int d^3\vec{v}\, v\, e^{-v^2/v_{th}^2}}{\int d^3\vec{v}\, e^{-v^2/v_{th}^2}}$$

So $\langle v_r \rangle = \sqrt{2} \langle v \rangle$

So, we have $\nu_c = \sqrt{2}\,\sigma n\langle v\rangle = \frac{1}{\tau_c}$

and $$\lambda_{mfp} = \langle v\rangle \tau_c = \frac{1}{\sqrt{2}\,\sigma n}$$

So we pick up a numerical factor – not a big deal (but many books like keeping this particular prefactor)

<u>Note</u>. We could have decided to define these quantities in terms of rms speeds:

$$\nu_c = \sigma n\langle v_r^2\rangle^{1/2} = \sqrt{2}\,\sigma n\langle v^2\rangle^{1/2}$$

$\langle \vec{v}_1\rangle\cdot\langle\vec{v}_2\rangle = 0$

because $$\langle v_r^2\rangle = \langle|\vec{v}_1-\vec{v}_2|^2\rangle = \langle v_1^2\rangle + \langle v_2^2\rangle - 2\langle\vec{v}_1\cdot\vec{v}_2\rangle$$
$$= 2\langle v^2\rangle$$

Then $\lambda_{mfp} = \langle v^2\rangle^{1/2}\tau_c = \frac{1}{\sqrt{2}\,\sigma n}$ again.