Statistical Physics

Second year physics course

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Problem Set 7

Some useful constants

Boltzmann's constant	k_B	$1.3807 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Stefan's constant	σ	$5.67 \times 10^{-8}\ \mathrm{Wm^{-2}K^{-4}}$
Proton rest mass	m_p	$1.6726 \times 10^{-27}\,\mathrm{kg}$
Bohr magneton	μ_B	$9.274 \times 10^{-24}\,\mathrm{JT^{-1}}$
Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
Standard molar volume		$22.414 \times 10^{-3}\,\mathrm{m^3\,mol^{-1}}$
Molar gas constant	R	$8.315\ \mathrm{J\,mol^{-1}\,K^{-1}}$
1 pascal (Pa)		$1\,\mathrm{N\,m^{-2}}$
1 standard atmosphere		$1.0132 \times 10^5\,\mathrm{Pa\,(N\,m^{-2})}$
1 bar (= 1000 mbar)		$10^5\,\mathrm{N\,m^{-2}}$

PROBLEM SET 7: Quantum Gases and Thermal Radiation

Note. Starred questions are not particularly harder than the unstarred ones, but you are struggling/runnibg out of time, you may skip (some of) them.

Quantum Gases

7.1 *Ultrarelativistic Quantum Gas.* Consider an ideal quantum gas (Bose or Fermi) in the ultrarelativistic limit.

(a) Find the equation that determines its chemical potential (implicitly) as a function of density n and temperature T.

(b) Calculate the energy U and grand potential Φ and hence prove that the equation of state can be written as

$$pV = \frac{1}{3}U$$

completely generally, regardless of whether the gas is in the classical limit, degenerate limit or in between.

(c) Consider an adiabatic process with the number of particles held fixed and show that

$$pV^{4/3} = \text{const}$$

for any temperature and density (not just in the classical limit).

Hint. Express entropy S and particle number N in the form $V^\alpha T^\beta f(\mu/T)$, where α and β are some powers and f some function of one argument. Hence argue what happens when S and N are held constant.

(d) Show that in the hot, dilute limit (large T, small n), $e^{\mu/k_BT} \ll 1$. Find the specific condition on n and T that must hold in order for the classical limit to be applicable. Hence derive the condition for the gas to cease to be classical and become degenerate. Estimate the minimum density for which an electron gas can be simultaneously degenerate and ultrarelativistic.

(e) Find the Fermi energy ε_F of an ultrarelativistic electron gas and show that when $k_BT \ll \varepsilon_F$, the heat capacity is

$$C_V = Nk_B\pi^2\frac{k_BT}{\varepsilon_F}.$$

Sketch the heat capacity of an ultrarelativistic electron gas as a function of temperature, from $T \ll \varepsilon_F/k_B$ to $T \gg \varepsilon_F/k_B$.

7.2 *Pair Production.* At relativistic temperatures, the number of particles can stop being a fixed number, with production and annihilation of electron-positron pairs providing the number of particles required for thermal equilibrium. The reaction is

$$e^+ + e^- \Leftrightarrow \text{photon(s)}.$$

(a) What is the condition of "chemical" equilibrium for this system?

(b) Assume that the numbers of electrons and positrons are the same (i.e., ignore the fact that there is ordinary matter and, therefore, a surplus of electrons). This allows you to assume that the situation is fully symmetric and the chemical potentials of electrons and positrons are the same. What are they equal to? Hence calculate the densitiy of electrons and positrons $n^\pm$ as a function of temperature, assuming $k_BT \gg m_ec^2$. You will need to know that

$$\int_0^\infty \frac{dx\, x^2}{e^x+1} = \frac{3}{2}\zeta(3), \quad \zeta(3) \approx 1.202$$

(see, e.g., Landau & Lifshitz §58 for the derivation of this formula).

(c) To confirm an *a priori* assumption you made in (b), show that at ultrarelativistic temperatures, the density of electrons and positrons you have obtained will always be larger than the density of electrons in ordinary matter. This will require you to come up with a simple way of estimating the upper bound for the latter.

(d*) Now consider the non-relativistic case, $k_BT \ll m_ec^2$, and assume that temperature is also low enough for the classical (non-degenerate) limit to apply. Let the density of electrons in matter, without the pair production, be n_0. Show that the density of positrons due to spontaneous pair production, in equilibrium, is exponentially small:

$$n^+ \approx \frac{4}{n_0}\left(\frac{m_ek_BT}{2\pi\hbar^2}\right)^3 e^{-2m_ec^2/k_BT}.$$

Hint. Use the law of mass action. Note that you can no longer assume that pairs are more numerous than ordinary electrons. Don't forget to reflect in your calculation the fact that the energy cost of producing an electron or a positron is m_ec^2.

7.3 *Paramagnetism of a Degenerate Electron Gas (Pauli Magnetism).* Consider a fully degenerate electron gas in a weak magnetic field. Since the electrons have two spin states (up and down), take the energy levels to be

$$\varepsilon(\mathbf{k}) = \frac{\hbar^2k^2}{2m} \pm \mu_BB,$$

where $\mu_B = e\hbar/2m_ec$ is the Bohr magneton (in cgs-Gauss units). Assume the field to be sufficiently weak so that $\mu_BB \ll \varepsilon_F$.

(a) Show that the magnetic susceptibility of this system is

$$\chi = \left(\frac{\partial M}{\partial B}\right)_{B=0} = \frac{3^{1/3}}{4\pi^{4/3}}\frac{e^2}{m_ec^2}\,n^{1/3},$$

where M is the magnetisation (total magnetic moment per unit volume) and n density (the first equality above is the definition of χ, the second is the answer you should get).

Hint. Express M in terms of the grand potential Φ. Then use the fact that energy enters the Fermi statistics in combination $\varepsilon - \mu$ with the chemical potential μ. Therefore, in order to calculate the individual contributions from the spin-up and spin-down states to the integrals over single-particle states, we can use the unmagnetised formulae with μ replaced by $\mu \pm \mu_B B$, viz., the grand potential, for example, is

$$\Phi(\mu, B) = \frac{1}{2}\,\Phi_0(\mu + \mu_B B) + \frac{1}{2}\,\Phi_0(\mu - \mu_B B),$$

where $\Phi_0(\mu) = \Phi(\mu, B = 0)$ is the grand potential in the unmagnetised case. Make sure to take full advantage of the fact that $\mu_B B \ll \varepsilon_F$.

(b) Show that in the classical (non-degenerate) limit, the above method recovers Curie's law. Sketch χ as a function of T, from very low to very high temperature.

(c*) Show that at $T \ll \varepsilon/k_B$, the finite-temperature correction to χ is quadratic in T and negative (i.e., χ goes down as T increases).

7.4 *Degenerate Bose Gas in 2D.*

(a) Show that Bose condensation does not occur in 2D.

(b*) Calculate the chemical potential as a function of n and T in the limit of small T. Sketch $\mu(T)$ from small to large T.

(c*) Show that the heat capacity (at constant area) is $C \propto T$ at low temperatures and sketch $C(T)$ from small to large T.

7.5 *Paramagnetism of a Degenerate Bose Gas.* Consider bosons with spin 1 in a weak magnetic field, with energy levels

$$\varepsilon(\mathbf{k}) = \frac{\hbar^2 k^2}{2m} - 2\mu_B s_z B, \quad s_z = -1, 0, 1,$$

where $\mu_B = e\hbar/2m_e c$ is the Bohr magneton (in cgs-Gauss units).

(a) Derive an expression for the magnetic susceptibility of this system. Show that Curie's law ($\chi \propto 1/T$) is recovered in the classical limit.

(b) What happens to $\chi(T)$ as the temperature tends to the critical Bose-Einstein condensation temperature from above ($T \to T_c + 0$)? Sketch $\chi(T)$.

(c) At $T < T_c$ and for a given B, which quantum state will be macroscopically occupied? Taking $B \to +0$ (i.e., infinitesimally small), calculate the spontaneous magnetisation of the system, $M_0(n, T) = \lim_{B\to 0} M(n, T, B)$, as a function of n and T. Explain why the magnetisation is non-zero even though B is vanishingly small. Does the result of (b) make sense in view of what you have found?

7.6 (*) *Bose and Fermi Statistics from Maximum Entropy.* It turns out that even out of equilibrium, the entropy of Fermi and Bose gases satisfies

$$S = -k_B \sum_i \left[\overline{n}_i \ln \overline{n}_i \pm (1 \mp \overline{n}_i) \ln(1 \mp \overline{n}_i)\right],$$

where $\overline{n}_i$ are mean occupation numbers of single-particle states i, the upper signs are for fermions and the lower ones for bosons (how to prove that this holds even out of equilibrium is explained in Landau & Lifshitz's book; the proof of the above formula *in* equilibrium is in the lecture notes). Considering a system with fixed average energy and number of particles, derive from the above the Fermi-Dirac and Bose-Einstein formulae for the mean occupation numbers in equilibrium.

Thermal Radiation

7.7 Thermal radiation can be treated thermodynamically as a gas with internal energy $U = u(T)V$ and pressure $p = u/3$. Show that

(i) the entropy density at constant temperature $s = 4p/T$,

(ii) the Gibbs function $G = 0$,

(iii) the chemical potential $\mu = 0$.

7.8 Outline the steps leading to the formula for the number of photons with angular frequencies between ω and $\omega + \mathrm{d}\omega$ in blackbody radiation at a temperature T:

$$n(\omega)d\omega = 2 \times \frac{V}{2\pi^2 c^3} \frac{\omega^2\,\mathrm{d}\omega}{\mathrm{e}^{\hbar\omega/k_\mathrm{B}T} - 1}.$$

Show that $n(\omega)$ has a peak at a frequency given by $\omega = 1.59 k_\mathrm{B}T/\hbar$. Show further that the spectral energy densities u_λ and u_ω peak at $\lambda_\mathrm{max} = hc/(4.97 k_\mathrm{B}T)$ and $\omega_\mathrm{max} = 2.82 k_\mathrm{B}T/\hbar$, respectively.

7.9 (i) The gas pressure at the centre of the Sun is 4×10^{11} atmospheres, and the temperature is $2 \times 10^7\,\mathrm{K}$. Estimate the radiation pressure and show that it is very small compared with the gas pressure.

(ii) The surface temperature of the Sun is 5,700 K, and the spectrum of radiation it emits has a maximum at a wavelength of 510 nm. Estimate the surface temperature of the North Star, for which the corresponding maximum is 350 nm.

(iii) Assuming the Sun (radius $6.955 \times 10^8\,\mathrm{m}$) emits radiation as a black body, calculate the solar power incident on a thin black plate of area $1\,\mathrm{m}^2$ facing the Sun and at a distance of 1 Astronomical Unit ($= 1.496 \times 10^{11}\,\mathrm{m}$) from it. Find the temperature of the plate given that it emits radiation from both surfaces. Neglect any radiation incident on the surface of the plate facing away from the Sun.