Recall that Lagrangian of scalar electrodynamics takes the following form in unitary gauge

$$\mathcal{L} = \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2} m_h^2 h^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 A_\mu A^\mu$$

$$+ m_A h A_\mu A^\mu + \frac{1}{2} h^2 A_\mu A^\mu + O(h^3, h^4)$$

The structure of the interactions of the Higgs field $h$ with the gauge field $A_\mu$ is fixed by gauge symmetry

To understand that gauge symmetry is indeed important, let us assume that instead of

$$\mathcal{L}_{int} = m_A h A_\mu A^\mu + \frac{1}{2} h^2 A_\mu A^\mu$$

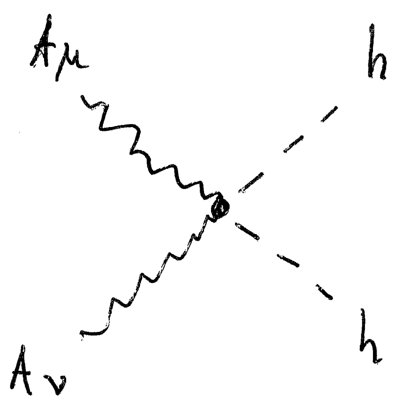
we would have only an interaction of the form

$$\mathcal{L}'_{int} = \frac{1}{2} h^2 A_\mu A^\mu$$

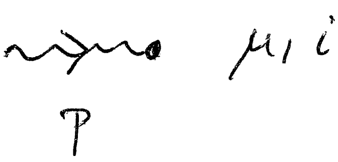
Now let us try to compute a physical observable that involves the new $\mathcal{L}'_{int}$. Let us look at the cross section for

$$AA \to hh$$

To calculate $\sigma(AA \to hh)$ we need the Feynman rules for the $h^2A^2$ vertex and the external gauge fields :




$$= \quad i\eta_{\mu\nu}$$




$$= \quad \varepsilon_i^{\mu}(p)$$

Here $\varepsilon_i^\mu(p)$ with $i = +, -, L$ denoting the three physical polarizations of the massive gauge field $A_\mu$

Why three polarizations?

Recall that the Euler-Lagrange equations of the free gauge field theory

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 A_\mu A^\nu$$

are

$$\partial_\mu F^{\mu\nu} + m_A^2 A^\nu = 0$$

Taking the derivative $\partial_\nu$ we see that

$$\partial_\nu \partial_\mu F^{\mu\nu} + m_A^2 \partial_\nu A^\nu = 0$$

which implies that a solution of the Euler-Lagrange equations also satisfies

$$\partial_\nu A^\nu = 0$$

i.e. the Lorenz gauge condition. This allows to eliminate one of the four degrees of freedom of $A_\mu$. Since the mass term

$$\frac{1}{2} m_A^2 A_\mu A^\mu$$

is not gauge invariant it is however not possible to gauge away another degree of freedom

The scattering amplitude for $AA \to hh$ for fixed polarizations $i,j$ takes the form

$$A_{ij} = i\, \varepsilon_i(p_1) \cdot \varepsilon_j(p_2)$$

To figure out if some of the polarizations give a more important contribution to the cross section, we need to know the explicit form of the $\varepsilon_i^\mu(p)$ vectors

We write

$$A^\mu(x) = \int \frac{d^4p}{(2\pi)^4}\, \varepsilon^\mu(p)\, e^{ipx}$$

and realize that $\partial_\nu A^\nu = 0$ in Fourier space just gives

$$p_\nu \varepsilon^\nu(p) = 0$$

Now we choose

$$p^\nu = (E, 0, 0, p)$$

with

$$p_\nu p^\nu = E^2 - p^2 = m_A^2$$

The transversal polarization $\varepsilon_\pm^\mu(p)$ can thus be taken to be

$$\varepsilon_+^\mu(p) = (0,1,0,0)$$

$$\varepsilon_-^\mu(p) = (0,0,1,0)$$

They satisfy

$$p_\mu \varepsilon_\pm^\mu(p) = 0$$

$$(\varepsilon_\pm^\mu)^2 = -1$$

In the longitudinal case we choose

$$\varepsilon_L^\mu(p) = \frac{1}{m_A}(p, 0, 0, E)$$

which fulfils

$$p_\mu \varepsilon_L^\mu(p) = \frac{1}{m_A}(Ep - pE) = 0$$

$$(\varepsilon_L^\mu)^2 = \frac{1}{m_A^2}(p^2 - E^2) = -1$$

The important point to notice now is that in the high-energy limit

$$E \gg m_A \, , m_h$$

one has

$$p^\mu \simeq E(1,0,0,1)$$

$$\varepsilon_L^\mu(p) \simeq \frac{E}{m_A}(1,0,0,1) = \frac{p^\mu}{m_A}$$

The longitudinal polarization vector hence grows linearly with the 4-momentum, while the transversal polarizations $\varepsilon_\pm^\mu(p)$ do not

Let us now look at the sum over amplitudes squares

$$\sum_{i,j=+,-,L} |\mathcal{A}_{ij}|^2$$

in the limit $E \gg m_A, m_h$

We get

$$\sum_{i,j=+,-,L} |\mathcal{A}_{ij}|^2 \simeq |i\, \varepsilon_L(p_1)\cdot\varepsilon_L(p_2)|^2$$

$$\simeq |i\, p_1\cdot p_2|^2 / m_A^2$$

In the centre-of-mass frame

$$p_1^\mu \simeq E(1,0,0,1) \quad , \quad p_2^\mu \simeq E(1,0,0,-1)$$

one thus has

$$\sum_{i,j=+,-,L} |\mathcal{A}_{ij}|^2 \simeq \frac{4E^4}{m_A^4}$$

The differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{(16\pi)^2 E^2} \sum_{i,j=+,-,L} |\mathcal{A}_{ij}|^2$$

is in the high-energy limit then given by

$$\frac{d\sigma}{d\Omega} \simeq \frac{1}{32\pi^2}\frac{E^2}{m_A{}^4}$$

and the total cross section is

$$\sigma \simeq \frac{1}{16\pi}\frac{E^2}{m_A{}^4}$$

This is an inacceptable result because the cross section should not grow with energy, because this violates unitarity

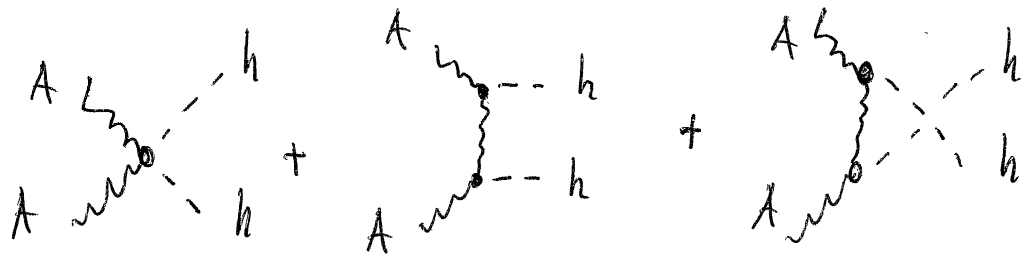
This calculation shows that gauge invariance is an utterly important feature

I leave it as an exercise to show that in the original theory of scalar electrodynamics with

$$\mathcal{L}_{int} = m_A \, h \, A_\mu A^\mu + \frac{1}{2} h^2 A_\mu A^\mu$$

the scattering amplitude of longitudinal gauge fields to two Higgses, $A_L A_L \to hh$, does not grow with energy

That uniterity is not violated has to do with the fact that in the presence of $\mathcal{L}_{int}$ one has not a single, but three diagrams




The latter two involve a $hA^2$ vertex which was not present in $\mathcal{L}_{int}$