M.Phys Option in Theoretical Physics: C6. Problem Sheet 6

1) A Green's function

$$\Delta(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\,\tilde{\Delta}(k)\ , \qquad (1)$$

of the Klein-Gordon equation is defined as a function satisfying $(\Box + m^2)\Delta(x) = -i\delta^{(4)}(x)$.
a) Show that the Fourier transform $\tilde{\Delta}$ of Δ is given by $\tilde{\Delta}(k) = i/(k^2 - m^2)$.
b) Insert the result from **a)** into (1) and discuss the various paths in the complex k_0 plane in relation to the two poles at $k_0 = \pm\sqrt{\mathbf{k}^2 + m^2}$ which can be chosen to carry out the k_0 integral.
c) Perform the k_0 integral for the paths in **b)** and show that for one choice of path $\Delta(x) = 0$ whenever $x_0 > 0$ and for another choice $\Delta(x) = 0$ whenever $x_0 < 0$. Interpret the physical relevance of these two Green's functions, as solutions to the Klein-Gordon equation with a delta function source.

2) Consider a real, free scalar field ϕ with mass m, evaluate the following time-order product of field operators $\int d^4x\, \langle 0|\, T\left(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi(x)^4\right)|0\rangle$, using Wick's theorem and draw the associated Feynman diagrams. Focus on the part without loops, Fourier transform $x_1, \ldots, x_4$ to $k_1, \ldots, k_4$ and express the result in terms of momentum-space Feynman propagators.

3) Consider a complex, free scalar field φ. Apply Wick's theorem to $T(\varphi(x_1)\varphi^\dagger(x_2))$ and prove the so-obtained relation explicitly. Do the same for $T(\varphi(x_1)\varphi(x_2))$.

4) Consider the decay of a particle with mass M into two particles with mass m.
a) Use the general formula for the decay rate Γ in terms of the matrix element $\mathcal{M}$ derived in the lecture (in fact, in the script I used the symbol $\mathcal{A}$ instead of $\mathcal{M}$ to denote amplitudes or matrix elements) to show that

$$\Gamma = \frac{1}{16\pi M}\sqrt{1 - \left(\frac{2m}{M}\right)^2}\,|\mathcal{M}|^2\ .$$

b) Two particle species A_1 and A_2 with masses m_1 and m_2 scatter in a process of the type $A_1 + A_2 \to A_1 + A_2$. For incoming momenta k_1, k_2 and outgoing momenta q_1, q_2 define the *Mandelstam variables* s, t, u by $s = (k_1 + k_2)^2$, $t = (k_1 - q_1)^2$, and $u = (k_1 - q_2)^2$. Show that $s + t + u = 2m_1^2 + 2m_2^2$.
c) Set $m = m_1 = m_2$ for simplicity. In the centre of mass frame, express the Mandelstam variables in terms of the total centre-of-mass (CM) energy $E_{\rm CM}$ and the scattering angle θ (and the mass m).
d) Show that the matrix element $\mathcal{M}$ for the scattering process in **b)** can always be written as a function of s and t.

5) Consider $2 \to 2$ scattering for the case where all four particles have identical masses.
a) Derive the explicit form of the 2-body phase-space element $\int d\Pi_2$ that has been introduced in the lecture.
b) Using your result calculate the angular differential cross section $(d\sigma/d\Omega)_{\text{CM}}$ for a generic $2 \to 2$ process in the CM frame. The solid angle $d\Omega$ is given by $d\phi\, d\cos\theta$ where $\theta \in [-\pi, \pi]$ is the polar scattering angle (with respect to the beam axis) and $\phi \in [0, 2\pi[$ the azimuthal scattering angle (around the beam axis). You should obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}(p_1, p_2 \to q_1, q_2)|^2}{64\pi^2 s}\,, \tag{2}$$

where $\mathcal{M}(p_1, p_2 \to q_1, q_2)$ represents the relevant scattering matrix element and $s = E_{\text{CM}}^2$ is the total CM energy squared.

6) A model with two real scalar fields ϕ_1 and ϕ_2 is described by the Lagrangian density $\mathcal{L} = \frac{1}{2}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 - M^2\phi_1^2 - m^2\phi_2^2 - \mu\phi_1\phi_2^2\right)$.
a) Derive the Feynman rule for the triplet vertex (*i.e.*, the vertex involving one ϕ_1 and two ϕ_2 fields) in this theory by computing the appropriate amputated Green's function.
b) Assume that $M > 2m$ and compute the rate Γ for the decay of a ϕ_1 particle into two ϕ_2 particles at leading order.
c) Compute the matrix element for scattering of two ϕ_2 particles into two ϕ_2 particles at leading order and express it in terms of Mandelstam variables.
d) Write the matrix element in **c)** in terms of the total CM energy E_{CM} and the scattering angle θ. Compute the differential cross section $(d\sigma/d\Omega)_{\text{CM}}$ for the process in **c)**, assuming, for simplicity, that $E_{\text{CM}} \gg m$.