M.Phys Option in Theoretical Physics: C6. Problem Sheet 4

1.) Consider the set $\mathrm{Sl}(2,\mathbb{C})$ of complex 2×2 matrices with determinant one.
a) Show that this set forms a group.
b) Compute the Lie-algebra of this group, show that the dimension of this algebra is 6 and write down a basis of generators using the Pauli matrices.
c) By making an appropriate choice for the generators and computing their commutation relations, show that the Lie algebra of $\mathrm{Sl}(2,\mathbb{C})$ is a representation of the Lorentz group Lie algebra.
d) As explained in Section 1.1 of the script "Elements of Classical Field Theory", representations of the Lorentz group Lie algebra are classified by two spins (j_+, j_-). Which pair of spins does the Lie algebra of $\mathrm{Sl}(2,\mathbb{C})$ corresponds to and why?

2.) The Lagrangian density $\mathcal{L} = \mathcal{L}(\partial_\mu\phi_a(x), \phi_a(x))$ for a set of fields $\phi_a = \phi_a(x)$ is assumed to be invariant under the infinitesimal transformation

$$\phi_a \to \phi_a - it^i (T_i)_a{}^b \phi_b \ ,$$

where T_i are the generators of a Lie algebra with commutators $[T_i, T_j] = if_{ij}{}^k T_k$ and t^i are the symmetry parameters.

Find the conserved currents $j_{i\mu}$ and the associated conserved charges Q_i for this symmetry.

3.) A model with a real scalar field σ and three other real scalar fields $\boldsymbol{\phi} = (\phi_a)$ is specified by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\phi_a\partial^\mu\phi_a\right) - V(\sigma,\boldsymbol{\phi}) \ , \quad V = \frac{m^2}{2}(\sigma^2 + \boldsymbol{\phi}^2) + \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\phi}^2)^2 \ .$$

a) Show that this Lagrangian is invariant under SO(4) acting on the four-dimensional vectors $(\sigma, \boldsymbol{\phi})$.
b) Show that infinitesimal SO(4) transformations of the fields can be written as

$$\sigma \to \sigma + \boldsymbol{\beta}\cdot\boldsymbol{\phi} \ , \quad \boldsymbol{\phi} \to \boldsymbol{\phi} + \boldsymbol{\alpha}\times\boldsymbol{\phi} - \boldsymbol{\beta}\sigma \ ,$$

for suitable defined small symmetry parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.
c) Find the six conserved currents and charges.
d) Analyze spontaneous breaking of the SO(4) symmetry in the case where $m^2 < 0$. In particular, find the vacua, determine the unbroken sub-group and the Goldstone modes. (Hint: For the two final tasks, choose a minimum of the potential for which $\boldsymbol{\phi} = 0$ and $\sigma \neq 0$.)

4.) a) Derive the energy-momentum tensor $T^{\mu\nu}$ from the Lagrangian formulation of the free Maxwell theory, using the general procedure explained in the lecture.
b) Given that $T^{\mu\nu}$ is conserved show that

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho K^{\rho\mu\nu}$$

is still conserved provided the tensor $K^{\rho\mu\nu}$ is anti-symmetric in its first two indices.
c) Choosing $K^{\rho\mu\nu} = F^{\mu\rho}A^\nu$ show that $\tilde{T}^{\mu\nu}$ is a symmetric tensor.

d) Write down the conserved charges associated to $\tilde{T}^{\mu\nu}$ and (by writing them in terms of $\mathbf{E}$ and $\mathbf{B}$) show that they yield the standard expressions of the electromagnetic energy and momentum densities.

5.) Consider a real scalar $\phi(t,x)$ field living on a two-dimensional space-time and defined on an interval $x \in [0, L]$ with *Dirichlet boundary conditions* $\phi(t,0) = \phi(t,L) = 0$.
a) Show that the (classical) positive- and negative-frequency solutions to the Klein-Gordon equation that also satisfy the boundary conditions have the form

$$\phi_n^{(\pm)}(t,x) = \frac{1}{\sqrt{\omega_n L}}\, e^{\pm i\omega_n t}\, \sin(k_n x) \ .$$

Give the expression for k_n in terms of L. How is ω_n related to k_n?
b) Now quantise the field $\phi(t,x)$, keeping in mind that momentum is discrete

$$\phi(t,x) = \sum_{n=1}^{\infty} \left(\phi_n^{(-)}(t,x)\, a_n + \phi_n^{(+)}(t,x)\, a_n^\dagger \right) \ ,$$

with the annihilation/creation operators satisfying $[a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0$ and $[a_n, a_m^\dagger] = \delta_{mn}$. Compute the vacuum expectation value $\langle 0|\mathcal{H}|0\rangle$ of the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\left[\dot{\phi}^2 + (\partial_x \phi)^2 + m^2\phi^2\right] \ .$$

Integrating your result over the interval $[0, L]$ and show that the total vacuum energy is

$$E_0(L) = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n \ .$$

c) Since this quantity is infinite, we need some form of regularisation in order to handle the divergence. Let us introduce an exponentially damping function $\exp(-\delta\omega_n)$ with $\delta > 0$ in the sum, and consider for simplicity the case of a massless field. Prove that in this case the vacuum energy can be written as

$$E_0(L,\delta) = \frac{\pi}{8L}\, \sinh^{-2}\left(\frac{\delta\pi}{2L}\right) \ .$$

Take the limit $\delta \to 0$ and determine the vacuum energy for the case when no boundary conditions are imposed. With all this at hand calculate the *Casimir force*, that is, the attractive force associated to the mismatch between the vacuum energy of the unbounded space and that of the theory on the interval.