M.Phys Option in Theoretical Physics: C6. Problem Sheet 2 HT2018

Question 9. A lattice model for non-ideal gas is defined as follows. The sites i of a lattice may be empty or occupied by at most one atom, and the variable n_i takes the values $n_i = 0$ and $n_i = 1$ in the two cases. There is an attractive interaction energy J between atoms that occupy neighbouring sites, and a chemical potential μ. The model Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle ij \rangle} n_i n_j - \mu \sum_i n_i \,, \tag{1}$$

where $\sum_{\langle ij \rangle}$ is a sum over neighbouring pairs of sites.
(a) Describe briefly how the *transfer matrix method* may be used to calculate the statistical-mechanical properties of one-dimensional lattice models with short range interactions. Illustrate your answer by explaining how the partition function for a one-dimensional version of the lattice gas, Eq. (1), defined on a lattice of N sites with periodic boundary conditions, may be evaluated using the matrix

$$\mathbf{T} = \begin{pmatrix} 1 & \mathrm{e}^{\beta\mu/2} \\ \mathrm{e}^{\beta\mu/2} & \mathrm{e}^{\beta(J+\mu)} \end{pmatrix} .$$

(b) Derive an expression for $\langle n_i \rangle$ in the limit $N \to \infty$, in terms of elements of the eigenvectors of this matrix.
(c) Show that

$$\langle n_i \rangle = \frac{1}{1 + \mathrm{e}^{-2\theta}} \,,$$

where

$$\sinh(\theta) = \exp(\beta J/2) \sinh(\beta[J + \mu]/2) \,.$$

Sketch $\langle n_i \rangle$ as a function of μ for $\beta J \gg 1$, and comment on the physical significance of your result.

Question 10. The one-dimensional 3-state Potts model is defined as follows. At lattice sites $i = 0, 1, \ldots, L$ "spin" variables σ_i take integer values $\sigma_i = 1, 2, 3$. The Hamiltonian is then given by

$$H = -J \sum_{i=0}^{L-1} \delta_{\sigma_i, \sigma_{i+1}} ,$$

where $\delta_{a,b}$ is the Kronecker delta, $J > 0$.
(a) What are the ground states and first excited states for this model?
(b) Write down the transfer matrix for (2). Derive an expression for the free energy per site f in the limit of large L in terms of the transfer matrix eigenvalues. Show that vectors of the form $(1, z, z^2)$ with $z^3 = 1$ are eigenvectors, and hence find the corresponding eigenvalues. Show that at temperature T (with $\beta = 1/k_B T$) and in the limit $L \to \infty$

$$f = -k_B T \ln\left(3 + e^{\beta J} - 1\right) .$$

(c) The boundary variable σ_0 is fixed in the state $\sigma_0 = 1$. Derive an expression (for large L), that the variable at site $\ell \gg 1$ is in the same state, in terms of the transfer matrix eigenvalues and eigenvectors. Show that your result has the form

$$\langle \delta_{\sigma_\ell, 1} \rangle = \frac{1}{3} + \frac{2}{3} e^{-\ell/\xi} .$$

How does ξ behave in the low and high temperature limits?

Question 11. Consider a one dimensional Ising model on an open chain with N sites, where N is odd. On all even sites a magnetic field $2h$ is applied, see Fig. 1. The energy is

$$E = -J \sum_{j=1}^{N-1} \sigma_j \sigma_{j+1} + 2h \sum_{j=1}^{(N-1)/2} \sigma_{2j} . \tag{2}$$

(a) Show that the partition function can be written in the form

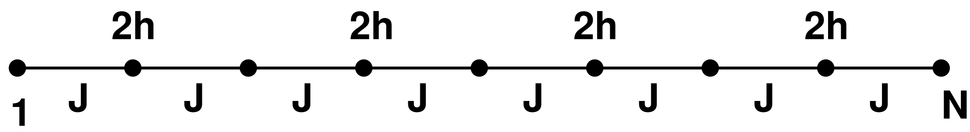

Figure 1: Open Ising chain with magnetic field applied to all even sites.

$$Z = \langle u|T^{(N-1)/2}|v\rangle ,$$

where T is an appropriately constructed transfer matrix, and $|u\rangle$ and $|v\rangle$ two dimensional vectors. Give explicit expressions for T, $|u\rangle$ and $|v\rangle$.
(b) Calculate Z for the case $h = 0$.

Question 12. Consider a Landau expansion of the free energy of the form

$$F = \frac{a}{2}m^2 + \frac{b}{4}m^4 + \frac{c}{6}m^6$$

with $c > 0$. Examine the phase diagram in the $a-b$ plane, and show that there is a line of critical transitions $a = 0$, $b > 0$ which joins a line of first order transitions $b = -4(ca/3)^{1/2}$ at a point $a = b = 0$ known as a tricritical point.

Supposing that a varies linearly with temperature and that b is independent of temperature, compare the value of the exponent β at the tricritical point with its value on the critical line.

From Yeomans, *Statistical Mechanics of Phase Transitions*

Question 13.
(a) Discuss how an order parameter may be used to characterise symmetry breaking at a phase transition.
(b) Argue that the uniaxial ferromagnet-paramagnet transition can be described by by a Landau free energy of the form

$$F = \int d^3\mathbf{r} \left[\frac{1}{2}|\nabla\phi(\mathbf{r})|^2 - h\phi(\mathbf{r}) + \alpha_2\phi^2(\mathbf{r}) + \alpha_3\phi^3(\mathbf{r}) + \alpha_4\phi^4(\mathbf{r})\right].$$

What can you say about α_4?
(c) What is the nature of the transition for $h = 0$ if $\alpha_3 \neq 0$? Explain your answer.
(d) Now assume that $\alpha_3 = h = 0$. Argue that close to the critical point

$$\alpha_2 = At\ , \quad t = \frac{T - T_c}{T_c} \text{ and } A > 0.$$

(e) Derive the equation characterizing the saddle point solution for $\alpha_3 = h = 0$. What are the configurations ϕ with the lowest free energy for $h = 0$, at $T > T_c$ and at $T < T_c$? Why are these $\mathbf{r}$ independent?
(f) Now consider more general solutions to the saddle point equation in the low-temperature phase. With suitable boundary conditions the saddle point solutions for the order parameter are functions of x only, i.e. $\phi = \phi(x)$. Show that in this case

$$E = \frac{1}{2}\left[\frac{d\phi(x)}{dx}\right]^2 - \alpha_2\phi^2 - \alpha_4\phi^4$$

is independent of x. Construct a solution $\phi(x)$ such that

$$\lim_{x\to\infty}\phi(x) = \phi_1\ , \quad \lim_{x\to-\infty}\phi(x) = \phi_2,$$

where $\phi_{1,2}$ are the solutions found in (d). Hint: determine E for such solutions first.

Question 14. A system with a real, two-component order parameter $(\phi_1(\mathbf{r}), \phi_2(\mathbf{r}))$ has a free energy

$$F = \int \mathrm{d}^d\mathbf{r} \left[\frac{1}{2}|\nabla\phi_1(\mathbf{r})|^2 + \frac{1}{2}|\nabla\phi_2(\mathbf{r})|^2 - \frac{1}{2}\left(\phi_1^2(\mathbf{r}) + \phi_2^2(\mathbf{r})\right) + \frac{1}{4}\left(\phi_1^2(\mathbf{r}) + \phi_2^2(\mathbf{r})\right)^2 \right] .$$

Find the order-parameter values Φ_1, Φ_2 that minimise this free energy. Consider small fluctuations around such state, with $(\phi_1(\mathbf{r}), \phi_2(\mathbf{r})) = (\Phi_1 + \varphi_1(\mathbf{r}), \Phi_2 + \varphi_2(\mathbf{r}))$ and expand F to second order in φ.

Assuming that the statistical weight of thermal fluctuations is proportional to $\exp(-F)$, calculate approximately the correlation function

$$\langle \varphi_1(\mathbf{r})\varphi_1(\mathbf{0}) + \varphi_2(\mathbf{r})\varphi_2(\mathbf{0}) \rangle$$

by evaluating a Gaussian functional integral. How does your result depend on the dimensionality d of the system?