M.Phys Option in Theoretical Physics 2017/18: C6. Problem Sheet 1

1.) Consider the set $\mathrm{Sl}(2,\mathbb{C})$ of complex 2×2 matrices with determinant one.
a) Show that this set forms a group.
b) Compute the Lie-algebra of this group, show that the dimension of this algebra is 6 and write down a basis of generators using the Pauli matrices.
c) By making an appropriate choice for the generators and computing their commutation relations, show that the Lie algebra of $\mathrm{Sl}(2,\mathbb{C})$ is a representation of the Lorentz group Lie algebra.
d) As explained in Section 1.3 of the notes "Elements of Classical Field Theory", representations of the Lorentz group Lie algebra are classified by two spins (j_+, j_-). Which pair of spins does the Lie algebra of $\mathrm{Sl}(2,\mathbb{C})$ corresponds to and why?

2.) The Lagrangian density $\mathcal{L} = \mathcal{L}(\partial_\mu\phi_a(x), \phi_a(x))$ for a set of fields $\phi_a = \phi_a(x)$ is assumed to be invariant under the infinitesimal transformation

$$\phi_a \to \phi_a - it^i (T_i)_a{}^b \phi_b \ ,$$

where T_i are the generators of a Lie algebra with commutators $[T_i, T_j] = i f_{ij}{}^k T_k$ and t^i are the symmetry parameters.

Find the conserved currents $j_{i\mu}$ and the associated conserved charges Q_i for this symmetry.

3.) A model with a real scalar field σ and three other real scalar fields $\boldsymbol{\phi} = (\phi_a)$ is specified by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\phi_a\partial^\mu\phi_a\right) - V(\sigma, \boldsymbol{\phi}) \ , \quad V = \frac{m^2}{2}(\sigma^2 + \boldsymbol{\phi}^2) + \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\phi}^2)^2 \ .$$

a) Show that this Lagrangian is invariant under SO(4) acting on the four-dimensional vectors $(\sigma, \boldsymbol{\phi})$.
b) Show that infinitesimal SO(4) transformations of the fields can be written as

$$\sigma \to \sigma + \boldsymbol{\beta}\cdot\boldsymbol{\phi} \ , \quad \boldsymbol{\phi} \to \boldsymbol{\phi} + \boldsymbol{\alpha}\times\boldsymbol{\phi} - \boldsymbol{\beta}\sigma \ ,$$

for suitable defined small symmetry parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.
c) Find the six conserved currents and charges.
d) Analyze spontaneous breaking of the SO(4) symmetry in the case where $m^2 < 0$. In particular, find the vacua, determine the unbroken sub-group and the Goldstone modes. (Hint: For the two final tasks, choose a minimum of the potential for which $\boldsymbol{\phi} = 0$ and $\sigma \neq 0$.)

4.) a) Derive the energy-momentum tensor $T^{\mu\nu}$ from the Lagrangian formulation of the free Maxwell theory, using the general procedure explained in the lecture.
b) Given that $T^{\mu\nu}$ is conserved show that

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho K^{\rho\mu\nu}$$

is still conserved provided the tensor $K^{\rho\mu\nu}$ is anti-symmetric in its first two indices.
c) Choosing $K^{\rho\mu\nu} = F^{\mu\rho}A^\nu$ show that $\tilde{T}^{\mu\nu}$ is a symmetric tensor.

d) Write down the conserved charges associated to $\tilde{T}^{\mu\nu}$ and (by writing them in terms of $\mathbf{E}$ and $\mathbf{B}$) show that they yield the standard expressions of the electromagnetic energy and momentum densities.

5.) The Lagrangian for a free complex scalar field reads

$$\mathcal{L}_\varphi = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi\,.$$

You know that this Lagrangian is invariant under a global $U(1)$ transformation,

$$\varphi \to e^{i\alpha}\, \varphi\,.$$

a) Does it stay invariant, if this symmetry gets promoted to a *local* symmetry, that is, $\alpha \to e\,\alpha(x)$, where e is just a universal constant and $\alpha(x)$ a function of space-time?
b) If you now add a vector field A_μ to the Lagrangian with a coupling

$$\mathcal{L}_{A\varphi} = i\lambda \left[\varphi^* (\partial_\mu \varphi) - \varphi(\partial_\mu \varphi^*)\right] A^\mu + \lambda^2\, (A_\mu \varphi^*)(A^\mu \varphi)\,,$$

how does the vector field have to transform under the local $U(1)$, if the Lagrangian $\mathcal{L}_\varphi + \mathcal{L}_{A\varphi}$ should remain invariant under phase redefinitions and what is the coupling constant λ?
c) Compute the Noether current for this local symmetry. Add a kinetic term for the vector field to the Lagrangian,

$$\mathcal{L}_A = -\frac{1}{4}\, F_{\mu\nu} F^{\mu\nu}\,,$$

and derive the equations of motion for the field A_μ considering the full Lagrangian $\mathcal{L}_\varphi + \mathcal{L}_{A\varphi} + \mathcal{L}_A$.