Local and non-local observables in $\mathcal{N} = 4$ *SYM*

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We will be interested in different observables of planar $\mathcal{N} = 4$ super Yang-Mills.

- A toy model for QCD: It can give non trivial information about QCD but at the same time is more tractable.
  - Perturbative computations are much simpler.
  - The strong coupling regime can be studied through a weakly coupled string sigma model.
- Impressive developments over the last few years. Unexpected structures, dualities and symmetries in many observables:
  - Wilson loops.
  - Scattering amplitudes.
  - Correlation functions.
  - Mixtures of them.

$\mathcal{N} = 4$ Super-Yang Mills (MSYM)

- Most symmetric four dimensional (gauge) quantum field theory.
- $SU(N)$ gauge group $\rightarrow$ fixed Lagrangian.
- Conformal symmetry: $SO(2,4)$.
- All particles are massless and in the adjoint representation:
  - A vector field (gluon/gauge field): A_μ.
  - Four complex fermions: ψ_A.
  - Six real scalars: Φ^I, or three complex $Z = \Phi^1 + i\Phi^2$,...

$\mathcal{N} = 4$ Super-Yang Mills

- Parametrized by N and g_{YM}.
- We will (mostly) focus in the planar limit: $N \gg 1$, $\lambda = g_{YM}^2 N$ fixed:

$$A(g_{YM}, N) \to A(\lambda)$$

Powerful tool to understand this theory: The AdS/CFT duality!

AdS/CFT duality

AdS/CFT duality

Four dimensional maximally SUSY Yang-Mills (g_{YM}, N)	$\Leftrightarrow$	Type IIB string theory on $AdS_5 \times S^5$ (g_s, R)

$$\sqrt{\lambda} \equiv \sqrt{g_{YM}^2 N} = \frac{R^2}{\alpha'} \qquad \frac{1}{N} \approx g_s$$

$SO(2,4)$ conformal group $\leftrightarrow$ issommetry group of AdS_5

- The AdS/CFT is a very powerful computational tool!

Consider $F(\lambda)$:

- The gauge theory is only good/reliable for $\lambda \ll 1$, where we can use perturbation theory.

Gauge theory:

$$F(\lambda) = F^{(0)} + \lambda F^{(1)} + \lambda^2 F^{(2)} + \ldots$$

- Systematic way to compute these terms, but the complexity grows really fast!

What to do for large values of λ?

- Use AdS/CFT! (remember $R \approx \lambda^{1/4}$)

String theory:

$$F(\lambda) = \sqrt{\lambda}\,\tilde{F}^{(0)} + \tilde{F}^{(1)} + \frac{1}{\sqrt{\lambda}}\tilde{F}^{(2)} + \dots$$

Some geometrical computation!

- In $\mathcal{N} = 4$ SYM we have the luxury of the AdS/CFT duality.
- We can compute quantities of $\mathcal{N} = 4$ SYM at strong coupling by doing geometrical computations on AdS.

<u>Obstacle</u>:

Which string theory observable corresponds to a given gauge theory observable?

Wilson loops

For a loop $\mathcal{C}$ embedded in four dimensional space we define the loop operator:

$$W(\mathcal{C}) = \frac{1}{N} Tr\, \mathcal{P} \exp\left(i\, g_{YM} \oint_{\mathcal{C}} A_\mu dx^\mu \right)$$

- Very interesting observables in gauge theories:
  - Phase acquired by an infinitely massive quark in the fundamental representation moving along a loop.
  - An order parameter for confinement.

- For any closed loop: a large class of observables!
- e.g. in pure Yang-Mills, these operators and their products form a complete basis of gauge invariant operators.

Wilson loops in $\mathcal{N} = 4$ Super-Yang Mills

In $\mathcal{N} = 4$ Super-Yang Mills it is convenient to study slightly different Wilson loops:

$$W(\mathcal{C}) = \frac{1}{N} Tr\, \mathcal{P} \exp\left(i\, g_{YM} \oint_{\mathcal{C}} (A_\mu \dot{x}^\mu + |\dot{x}| \Phi_I \theta^I) ds \right)$$

- The super-symmetric version of the ordinary Wilson loop (and locally super-symmetric).
- We have a coupling to the scalars. θ^I $(I = 1, ..., 6)$ is a unit vector in R^6.

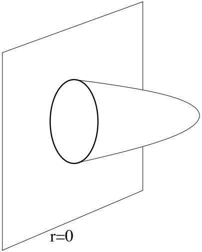
Weak coupling computation:

$$\langle W(\mathcal{C}) \rangle = 1 - \lambda \oint ds \oint ds' \dot{x}^\mu(s) \dot{x}^\nu(s') G_{\mu\nu}(x(s) - x(s')) + ... + \mathcal{O}(\lambda^2)$$

AdS/CFT: Expectation value of Wilson loops at strong coupling!

(Maldacena, Rey)

- We have a minimal area problem:

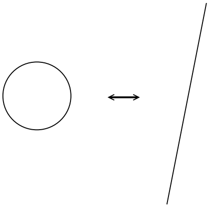


- $ds^2 = \frac{dx_{3+1}^2 + dr^2}{r^2}$
- We need to consider the minimal area ending (at $r = 0$) on the Wilson loop.

$$\langle W \rangle \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

- MSYM possesses powerful symmetries:
  - $SO(2,4)$ group of conformal symmetries.
  - Super-symmetry.
- In some cases, the answer is fixed by symmetries: some Wilson loops can be computed to all values of the coupling!

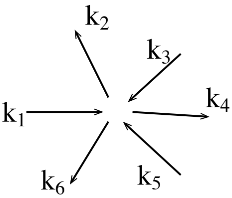
Inversion, $x^\mu \to \frac{x^\mu}{x^2}$: Circular W.L. $\leftrightarrow$ Straight line.



- The circular Wilson loop is known to all values of the coupling!

Scattering Amplitudes

Another interesting observable: Gluon scattering amplitudes.



$$= A_6(g_{YM}, N, k_1, \ldots)$$

<u>Motivation</u>: MSYM amplitudes can teach us about (and share many features with) QCD amplitudes but they are much more tractable.

- Large class of on-shell ($k_i^2 = 0$) observables.
- The kind of things you "measure".
- In principle computable by Feynman diagrams (good luck with that!).

- The amplitudes are IR divergent: use dimensional regularization.

$$D = 4 - 2\epsilon \to A_n^{(\ell)}(\epsilon) = 1/\epsilon^{2\ell} + ...$$

Exponentiation of IR divergences

$$A_n = \boxed{e^{S_{div}(\epsilon)}}\, e^{Finite}$$

Explicitly known!

- QCD divergences have very similar structure.

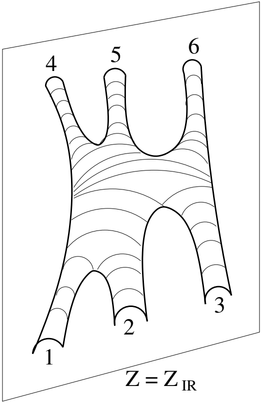
What about the helicities?

$$A(\pm, +, +, ..., +) = 0$$

- Simplest amplitudes, MHV: $A(-, -, +, ...+) \to$ function of kinematical invariants only.

AdS/CFT: Scattering amplitudes at strong coupling (L.F.A., Maldacena)

- Pretty complicated geometrical problem in AdS...



- $ds^2 = \frac{dy_{3+1}^2 + dz^2}{z^2}$
- Fixed hyperplane (brane) at $z = z_{IR}$ where open strings can end.
- Scatter these open strings.

- Classical solution very hard to find...

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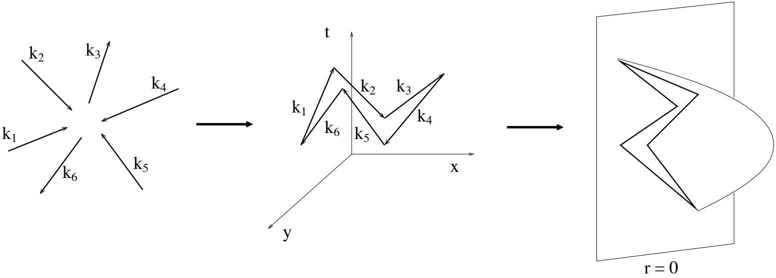
- Complicated change of coordinates $y \to x$ and $z \to 1/r$:

$$ds^2_{original} = \frac{dy^2_{3+1} + dz^2}{z^2} \;\to\; ds^2_{dual} = \frac{dx^2_{3+1} + dr^2}{r^2}$$

- The original AdS translates into a dual AdS but the boundary conditions simplify!

- Amplitudes at strong coupling → Minimal area in *AdS*!

$$ds^2 = \frac{dx_3^2 - dt^2 + dr^2}{r^2}$$



$$A_n \approx e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}, \qquad \lambda \gg 1$$

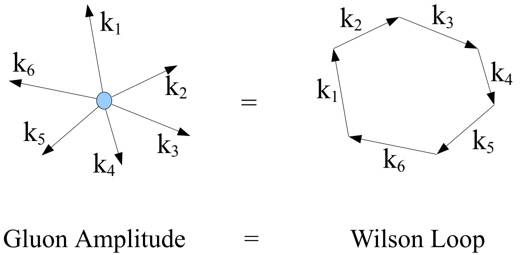
Surprises

- At strong coupling, exactly the expectation value of a polygonal light-like Wilson loop!
- There is a $SO(2,4)$ symmetry associated to the "dual" AdS space: dual conformal symmetry!

The duality with Wilson loops (and dual conformal symmetry) extends to all values of the coupling! (Henn, Korchemsky, Drummond, Sokatchev; Brandhuber, Heslop, Travaglini)

Amplitudes / Wilson loops duality

- For MHV scattering amplitudes we have:



- Very unexpected from the perturbative point of view!
- Led to analytic results for many amplitudes at two loops.
- Proved and extended to super-amplitudes by twistors techniques! (Caron-Huot; Mason, Skinner; Bullimore, Skinner). See talks by Bullimore and Adamo!

Amplitudes / Wilson loops duality

Surprising symmetry

Usual conformal symmetry of Wilson loops

$\Downarrow$

Dual conformal symmetry of scattering amplitudes!

- Nothing to do with the usual conformal symmetry.
- Fixes the amplitude up to a function of the cross-ratios.

$$\mathcal{A}_n = e^{f(\lambda) A_n^{one-loop}} e^{R(cross-ratios)}$$

- For $n = 4, 5$ we cannot construct any cross-ratios, so the symmetry is powerful enough to fix the amplitude for $n = 4, 5$!
- Symmetries can be exploited much further!

Correlation functions

New character in these developments: Correlation functions of gauge invariant local operators:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)...\mathcal{O}_n(x_n)\rangle$$

With

$$\mathcal{O}_1 = TrZD_+^s Z, \quad \mathcal{O}_2 = TrF_{\mu\nu}\psi_A X, \quad ...$$

- The natural observables in a conformal field theory.
- Natural generalization of two very important problems.

Generalization of the spectral problem

Spectral problem:

Two point functions of single trace local operators in $\mathcal{N}=4$ SYM.

$$\mathcal{O}_1 = \text{tr}ZZXX - \text{tr}ZXZX, \qquad \mathcal{O}_2 = \text{tr}ZZXX + \text{tr}ZXZX$$

Conformal symmetry: $\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{|x_{12}|^{2\Delta_i}}$

Spectral problem: Compute Δ_i to all values of the coupling!

$$\Delta_1 = 4, \qquad \Delta_2 = 4 + \frac{3}{\pi^2}\lambda + \ldots$$

AdS/CFT

Δ at strong coupling: Energy of a particular string configuration.

$$\Delta_1 = 4, \qquad \Delta_2 = 2\lambda^{1/4} + \ldots$$

What about three-point functions?

Conformal symmetry

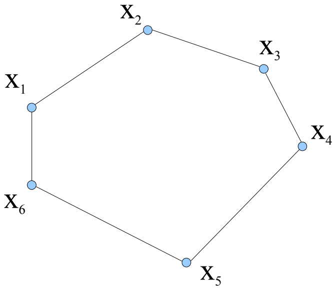
$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{13}^{\Delta_1+\Delta_3-\Delta_2}x_{23}^{\Delta_2+\Delta_3-\Delta_1}}$$

- We would like to compute $C_{123}(\lambda)$ to all values of the coupling constant.
- Knowing $\Delta_i(\lambda)$ plus $C_{ijk}(\lambda)$ we could compute any correlation function!

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle \to \sum_p \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_p\rangle\langle \mathcal{O}_p\mathcal{O}_3\mathcal{O}_4\rangle$$

Generalization of scattering amplitudes

- Also off-shell analogous of scattering amplitudes.
- Richer objects, depend on more cross-ratios: $\frac{(x_i - x_j)^2 (x_k - x_l)^2}{(x_i - x_k)^2 (x_j - x_l)^2}$



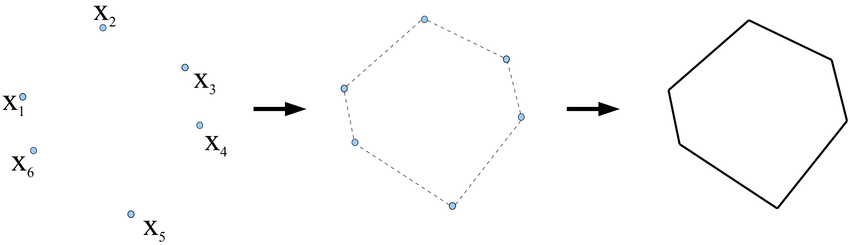
- Six-point amplitude $\rightarrow$ 3 cross-ratios.
- Six-point correlation function $\rightarrow$ 9 cross-ratios.

Generically finite, correlation functions can develop divergences:

- Usual OPE divergences when $x_i \to x_j$.
- Another divergence (light-cone OPE) when $(x_i - x_j)^2 \to 0$.

Interesting: Consecutive distances become null at the same rate:

$$x_{i,i+1}^2 = \epsilon^2 \to 0$$



The correlation function reproduces the null Wilson loop! [L.F.A, Eden,Korchemsky, Maldacena, Sokatchev]

Consider: $\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle$.

- $\mathcal{O} = Tr\phi^2(x)$ with ϕ: real scalar field in the adjoint representation

$$\lim_{x^2_{i,i+1}\to 0}\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle = \frac{1}{\prod_{i=1}^{n} x^2_{i,i+1}}\langle Tr_{adj}\,\mathcal{P}\exp\left(i\; g\oint_{\mathcal{C}_n} A_\mu dx^\mu\right)\rangle$$

Leading divergence, already in the free theory.

In the interacting theory also a finite correction, since the scalar field is color charged: approximated by a Wilson loop in this limit.

- $\mathcal{C}_n$: Polygonal null path of n edges.
- In the planar limit: $W_{adj}(\mathcal{C}_n) = W^2_{fund}(\mathcal{C}_n)$.

New duality: Correlation functions/Wilson loop

$$\lim_{x^2_{i,i+1}\to 0} \frac{\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle}{\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle_0} = \langle W_n \rangle^2_{fund}$$

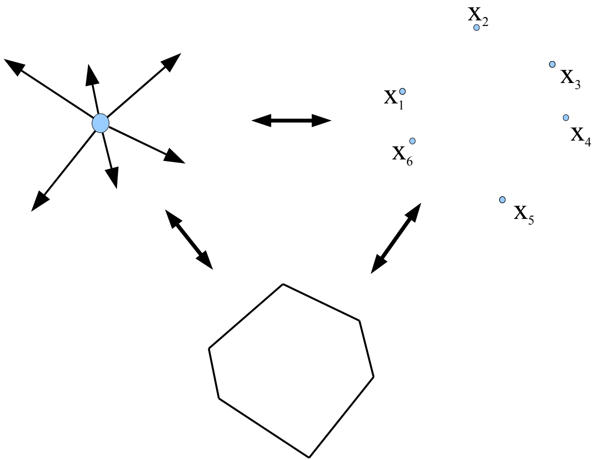
- Valid for a generic conformal field theory in any dimension!
- For $\mathcal{N}=4$ SYM has been extended to other local operators.

New tool to understand correlation functions:

- Non-trivial constraints in correlation functions.
- Led to new results for the four-point correlation function.

Triality of dualities

Amplitudes/Correlation Functions/Wilson Loops in $\mathcal{N}=4$ SYM



- Extended to general (not only MHV) amplitudes. [Eden, Heslop, Korchemsky, Sokatchev; Adamo, Bullimore, Mason, Skinner]

Yet another observable: Correlation functions of Wilson loops and local operators.

$$\langle W(\mathcal{C})\mathcal{O}(x)\rangle = \langle \mathcal{O}(x)\ Tr\ \mathcal{P}e^{ig\oint_{\mathcal{C}} A_\mu dx^\mu}\rangle$$

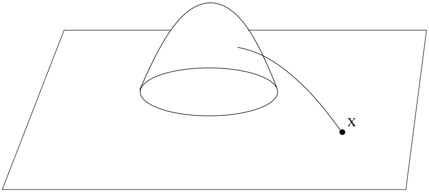
- They characterize the expansion of a Wilson loop in local operators:

$$W(\mathcal{C}) = \langle W(\mathcal{C})\rangle \sum_i c_i \mathcal{O}_i(x)$$

c_i can be found from $\langle W(\mathcal{C})\mathcal{O}_j\rangle$ and $\langle \mathcal{O}_i\mathcal{O}_j\rangle$.

- $\langle W(\mathcal{C})\mathcal{O}(x)\rangle$ can be computed at strong coupling! (for a large class of local operators) [Berenstein, Corrado, Fischler, Maldacena]

AdS/CFT: Computation at strong coupling, two ingredients.



- Classical solution (minimal surface) corresponding to $\langle W(\mathcal{C})\rangle$ (parametrized by X_{clas})
- A particular propagator $K^{\Delta}(x)$, which propagates from the point x in the boundary to the world-sheet of the classical solution.

$$\frac{\langle W\mathcal{O}^{\Delta}(x)\rangle}{\langle W\rangle}=\int d^2\zeta K^{\Delta}(x(\zeta)_{clas}-x, r(\zeta)_{clas})$$

$\langle W_n \mathcal{O}^\Delta(x) \rangle$: Correlation functions of null polygonal Wilson loops and local operators. [L.F.A., Buchbinder, Tseytlin; Tang, Roiban; Adamo]

- Makes connection with previous developments:

$$\lim_{x^2_{i,i+1}\to 0} \frac{\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\mathcal{O}(a)\rangle}{\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle} = \frac{\langle W_n \mathcal{O}(a)\rangle}{\langle W_n \rangle}$$

- Somewhere between a correlation function and a Wilson loop.
- Some properties of $\langle W_n \mathcal{O}(a)\rangle$ (e.g. behavior under conformal transformations) are easier to understand starting from the correlation function.
- Easiest example, $n = 4$: it depends on a single cross-ratio!

- Several dualities between local and non-local observables in $\mathcal{N}=4$ SYM.
- Drastic advances in the computation of such observables.

Questions:

- Can we use the integrability of planar $\mathcal{N}=4$ SYM to compute these observables?
- Integrability has been used to solve two related problems:
  - The spectral problem.
  - Scattering amplitudes/Wilson loops at strong coupling.
  - Some attempts for correlation functions.
- What about $\langle \mathcal{W}_n \mathcal{O}^{\Delta}(x) \rangle$? at strong coupling we know $\langle \mathcal{W}_n \rangle$ but without computing its classical solutions! can we use integrability?
- Can we extend this technology to other theories? e.g. QCD, theories of gravity...