Stokes Phenomena and Quantum Integrability in the Multi-cut Matrix Models

(and in Non-critical String/M Theory)

Hirotaka Irie

(National Center for Theoretical Sciences, Taiwan)

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Based on collaborations with

Chuan-Tsung Chan (THU) *and* **Chi-Hsien Yeh** (NTU)

General Motivation

How to define non-perturbatively complete string theory?

- String theory is defined by perturbation theory
- Despite of several candidates for non-perturbative formulations (SFT, Matrix theory…), we are still in the middle of the way:

  How to deal with the huge amount of string-theory vacua?
  Where is the true vacuum? Which are meta-stable vacua?
  How they decay into other vacua? How much is the decay rate?

- Stokes phenomenon is a bottom-up approach:

  How to *reconstruct* the non-perturbatively complete string theory
  from its perturbation theory?

- Here we study non-critical string theory. In particular, we will see that *the multi-cut matrix models* provide a nice toy model for this fundamental investigation.

Plan of the talk

1. Motivation for Stokes phenomenon (from physics)
   a) Perturbative knowledge from matrix models
   b) Spectral curves in the multi-cut matrix models
      (new feature related to Stokes phenomena)

2. Stokes phenomena and isomonodromy systems
   a) Introduction to Stokes phenomenon (of Airy function)
   b) General k x k ODE systems

3. Stokes phenomena in non-critical string theory
   a) Multi-cut boundary condition
   b) Quantum Integrability

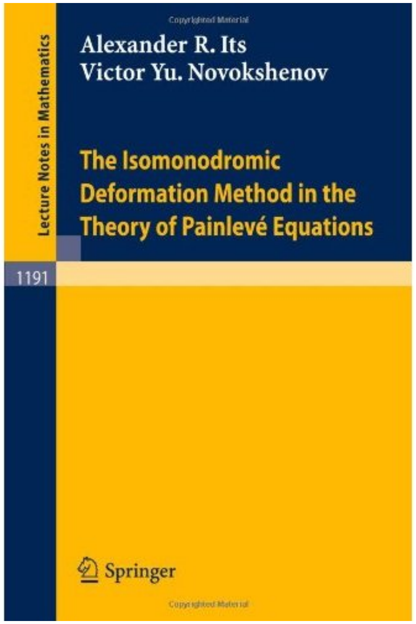
4. Summary and discussion

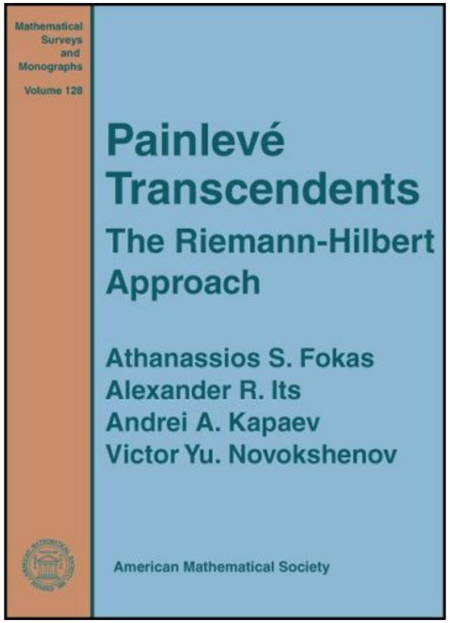
Main references

- Isomonodromy theory and Stokes phenomenon to matrix models (especially of Airy and Painlevé cases) [Moore '91]; [Maldacena–Moore–Seiberg–Shih '05]

- Isomonodromy theory, Stokes phenomenon and the Riemann–Hilbert (inverse monodromy) method (Painlevé cases: 2x2, Poincaré index r=2,3):

[Its–Novokshenov '91]; [Fokas-Its-Kapaev-Novokshenov'06]

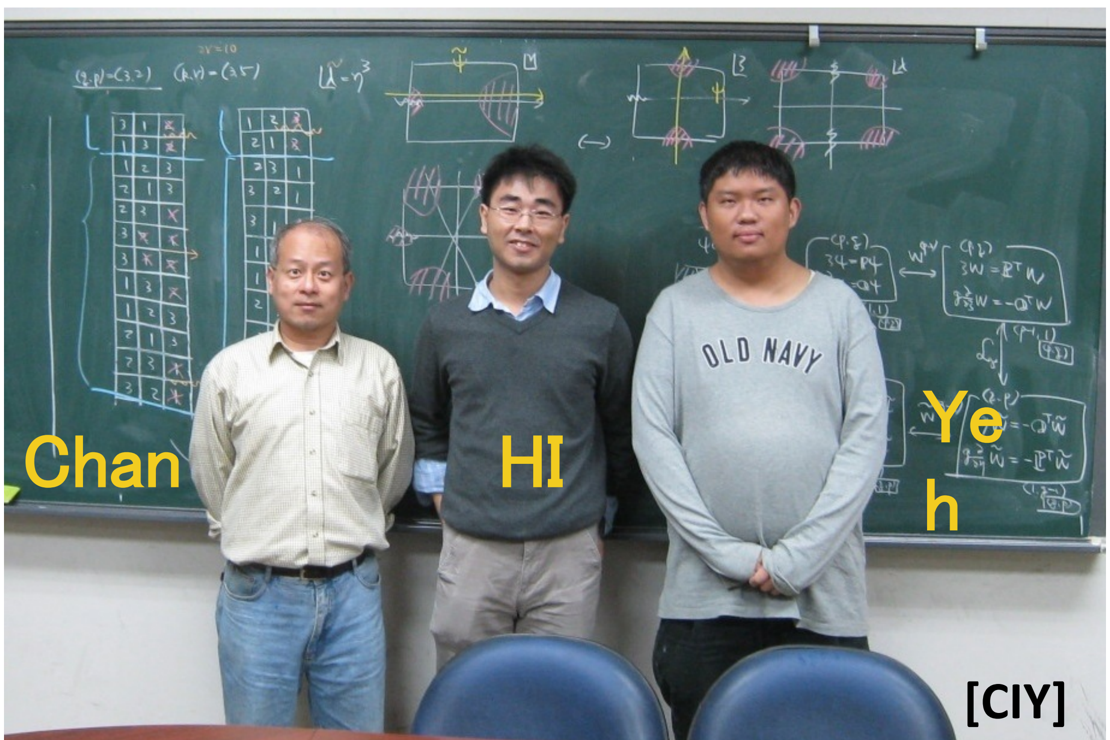


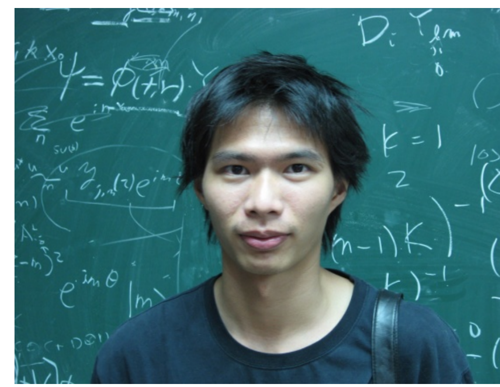


[FIKN]

Main references

- Stokes phenomena in general kxk isomonodromy systems corresponding to matrix models (general Poincaré index) [Chan-HI-Yeh 2 '10] ;[Chan-HI-Yeh 3 '11]; [Chan-HI-Yeh 4 '12, in preparation]
- Spectral curves in the multi-cut matrix models [Chan-HI-Shih-Yeh '09] ;[Chan-HI-Yeh 1 '10]





(S.-Y. Darren) Shih

[CISY]

1. Motivation for Stokes phenomenon

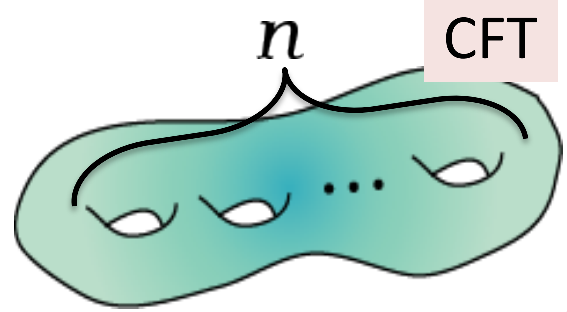
(from physics)

Ref) Spectral curves in the multi-cut matrix models:
[CISY '09] [CIY1 '10]

Perturbative knowledge from matrix models

(Non-critical) String theory

$$\mathcal{F} = \ln Z = \sum_{\substack{\text{possible WS} \\ (\text{genus: } n=0,1,\cdots)}} g^{2n-2}$$

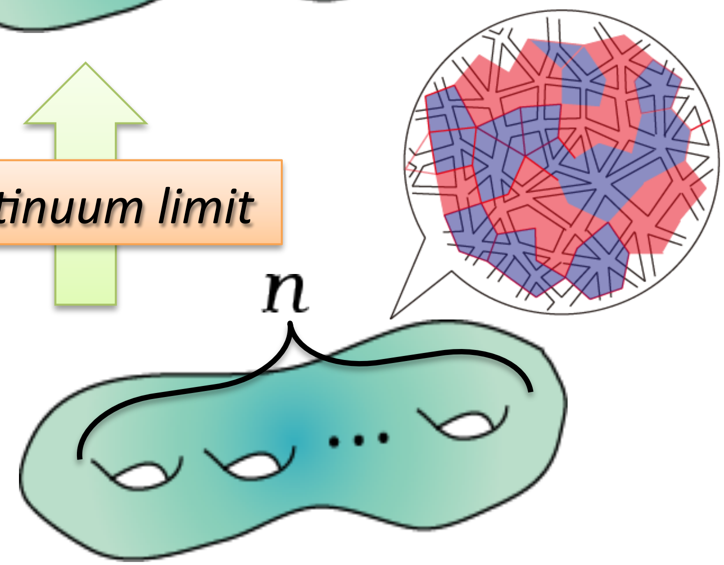


Continuum limit

Large N expansion of matrix models

$$Z_{\mathrm{MM}} = \int_{\text{N x N matrices}} dM e^{-N\mathrm{tr}V(M)}$$

$$\mathcal{F} = \ln Z_{MM} = \sum_{\substack{\text{Feynman Graph: } G \\ (\text{genus: } n=0,1,\cdots)}} N^{2-2n}$$



Triangulation (Lattice Gravity)

$g = N^{-1}$ *(Large N expansion ⇔ Perturbation theory of string coupling g)*

Matrix models know the world **beyond the perturbation theory**

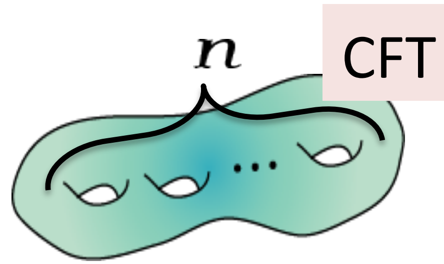
Non-perturbative corrections

D-instanton Chemical Potential

$$\mathcal{F} = \ln \mathcal{Z} \simeq \underbrace{\sum_{n=0}^{\infty} g^{2n-2}\mathcal{F}_n}_{\text{perturbative corrections}} + \underbrace{\sum_{I} \theta_I \exp\Big[\sum_{n=0}^{\infty} g^{2n+b_I-2}\mathcal{F}_n^{(I)}\Big]}_{\text{non-perturbative (instanton) corrections}}$$

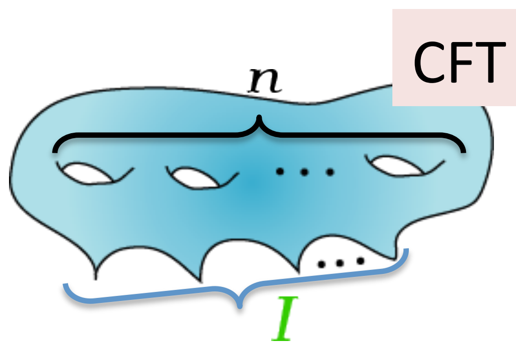
1. Perturbative amplitudes of WS_n:

$\mathcal{F}_n \quad \leftrightarrow$



2. Non-perturbative amplitudes are D-instantons! **[Shenker '90, Polchinski '94]**

$, \mathcal{F}_n^{(I)} \quad \leftrightarrow$



WS with Boundaries
= **open string theory**

3. The overall weight ϑ's (=Chemical Potentials) are out of the perturbation theory

Let's see more from the matrix-model viewpoints

Spectral curve

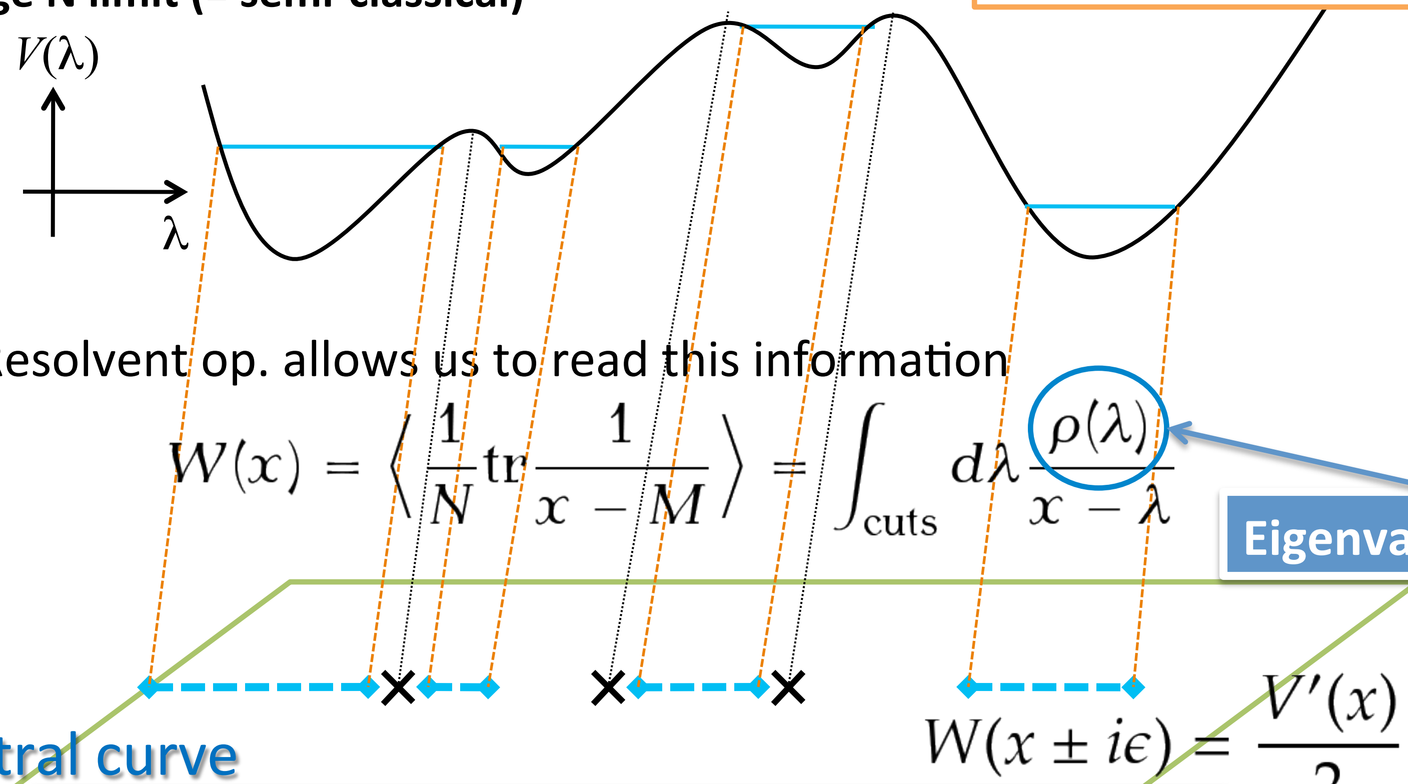
Resolvent: $$W(x) = \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{x-M} \right\rangle$$

Diagonalization: $$U^{\dagger}MU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$$

$$\mathcal{Z} = \int dM\, e^{-N\mathrm{tr}V(M)} \quad \Rightarrow \quad \mathcal{Z} = \int d^N\lambda \prod_{i>j}(\lambda_i - \lambda_j)^2 e^{-N\sum_i V(\lambda_i)}$$

N-body problem in the potential V

In Large N limit (= semi-classical)



The Resolvent op. allows us to read this information

$$W(x) = \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{x-M} \right\rangle = \int_{\mathrm{cuts}} d\lambda \frac{\rho(\lambda)}{x-\lambda}$$

Eigenvalue density

$$W(x \pm i\epsilon) = \frac{V'(x)}{2} \mp \pi i \rho(x)$$

Position of Cuts = Position of Eigenvalues

Why is it important?

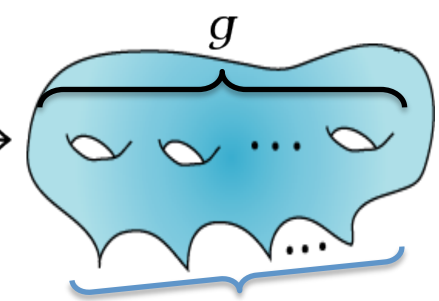
Spectral curve ⇔ Perturbative string theory

Perturbative correlators

$$W_n(z_1,z_2,\cdots,z_n) \equiv \Big\langle \prod_{j=1}^{n} \frac{1}{N}\mathrm{tr}\frac{1}{z_j - X}\Big\rangle_c$$

$$\simeq \sum_{g=0}^{\infty} N^{2-2g-n} W_n^{(g)}(z_1,\cdots,z_n)$$

$$W_n^{(g)}(J) \longleftrightarrow$$



$$J = (z_1, z_2, \cdots, z_n)$$

are all obtained recursively from the resolvent

Topological Recursions [Eynard'04, Eynard-Orantin '07]

$$W_{n+1}^{(g)} = \sum_i \mathop{\mathrm{Res}}_{z\to a_i} K(z_0,z)\Big[W_{n+2}^{(g-1)}(z,\bar{z},J) + \sum_{h=0}^{g}\sum_{I\subset J} W_{1+|I|}^{(h)}(z,I)W_{1+n-|I|}^{(g-h)}(\bar{z},J\setminus I)\Big]$$

Input: $W_1^{(0)}(z) = \lim_{N\to\infty} W_1(z)$, $K(z_0,z) \sim W_2^{(0)}(z_0,z)$:Bergman Kernel

Therefore, we symbolically write the free energy as

$$\mathcal{F}_{\mathrm{pert}}(\mathcal{C}) = \ln \mathcal{Z}_{\mathrm{pert}}(\mathcal{C}) = \sum_{n=0}^{\infty} g^{2n-2}\mathcal{F}_n(\mathcal{C}) \qquad (\mathcal{C}:\ \text{spectral curve})$$

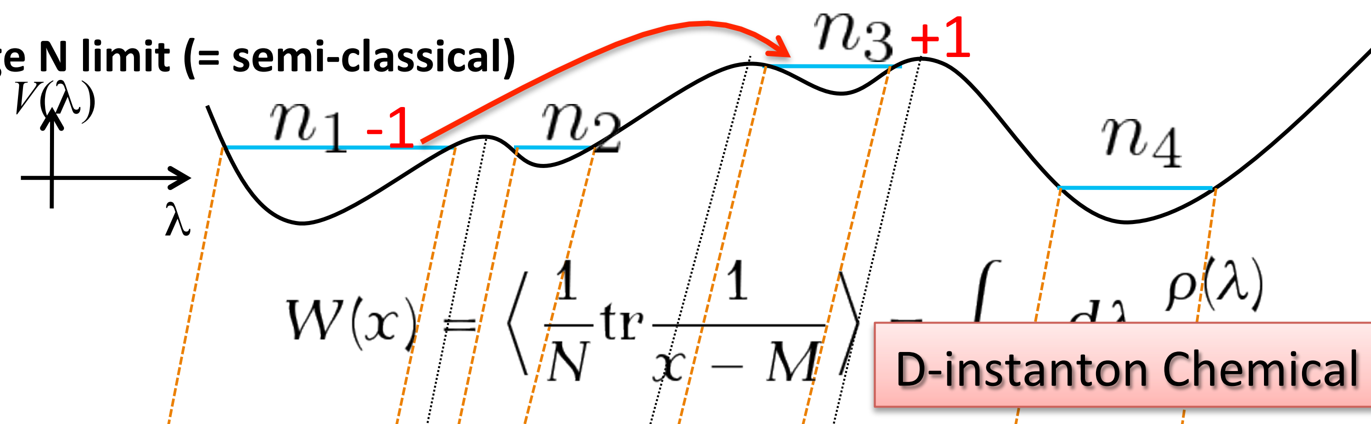
Why is it important?

Spectral curve ⇔ Perturbative string theory

Non-perturbative corrections

$$n_1 + n_2 + n_3 + n_4 = N$$

In Large N limit (= semi-classical)



$$W(x) = \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{x-M}\right\rangle = \int d\lambda \, \rho(\lambda)$$

D-instanton Chemical Potential

$$\mathcal{F} = \ln \mathcal{Z} \simeq \sum_{n=0}^{\infty} g^{2n-2}\mathcal{F}_n + \sum_I \theta_I \exp\left[\sum_{n=0}^{\infty} g^{2n+b_I-2}\mathcal{F}_n^{(I)}\right]$$

Non-perturbative partition functions: [Eynard '08, Eynard-Marino '08]

$$\mathcal{Z}(\mathcal{C}) = \underbrace{\sum_{n_1+\cdots+n_K=N}}_{\text{Summation over all the possible configurations}} \underbrace{\theta_1^{n_1}\cdots\theta_K^{n_K}}_{\text{with some free parameters}} e^{\mathcal{F}_{\mathrm{pert}}(\mathcal{C}_{n_1,\cdots,c_K})}$$

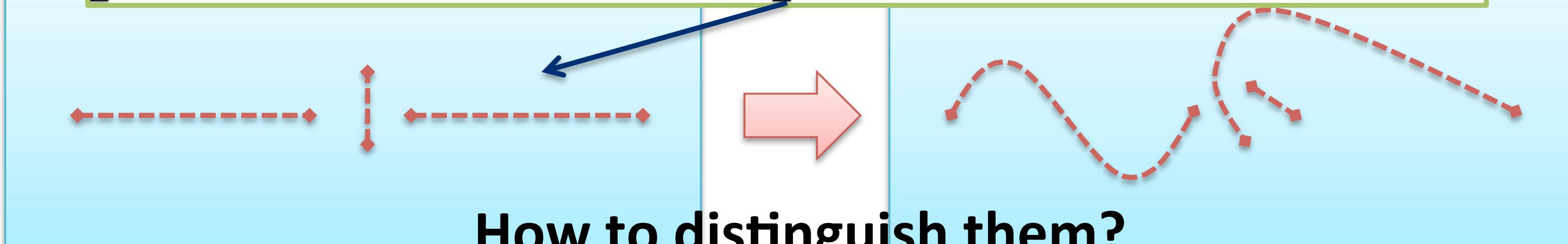
with some free parameters

Summation over all the possible configurations

What is the geometric meaning of the D-instanton chemical potentials?

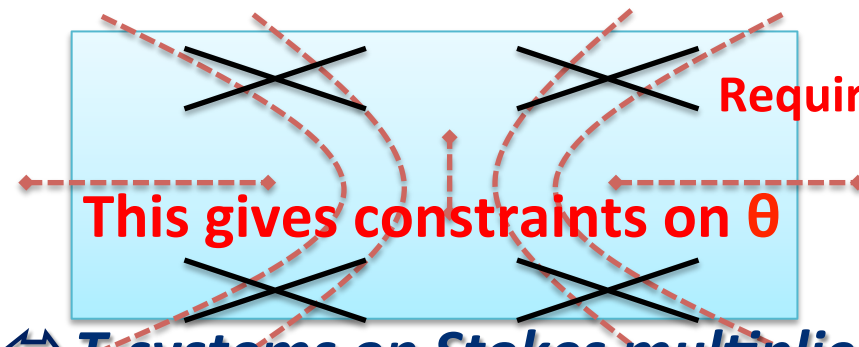
the Position of "Eigenvalue" Cuts **[CIY 2 '10]**

$$[W(x+i\epsilon)-W(x-i\epsilon)]\,dx=-2\pi i\rho(x)dx\in i\mathbb{R}$$



How to distinguish them?

But, we can also add infinitely long cuts



Later

⇔ T-systems on Stokes multipliers

"Physical cuts" as "Stokes lines of ODE" → ***Stokes phenomenon!***

$$\mathrm{Re}[\varphi(x)-\varphi'(x)]=0\ \ (\Leftrightarrow \rho(x)dx\in\mathbb{R})$$

From the Inverse monodromy (Riemann-Hilbert) problem [FIKN]

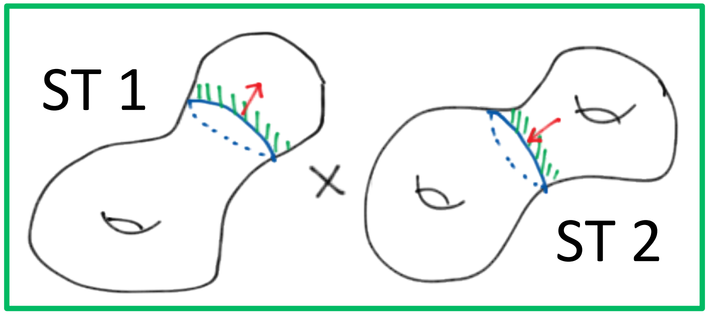
ϑ_I ≈ Stokes multipliers s_{l,I,j}

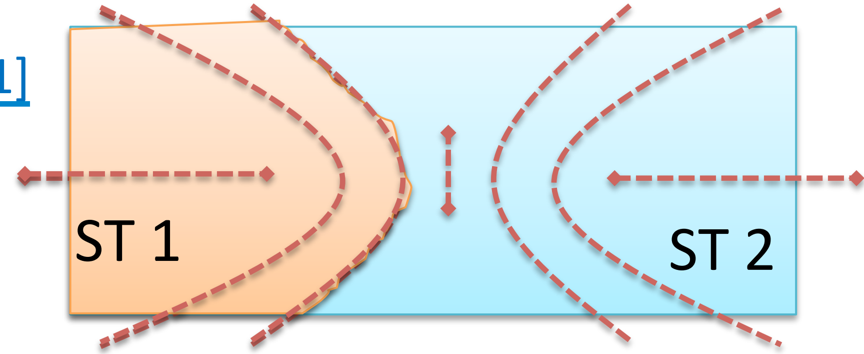
Why this is interesting?

The multi-cut extension [Crinkovic-Moore ‘91];[Fukuma-HI ‘06];[HI ‘09] !

1) Different string theories (ST) in spacetime [CIY 1 ‘10];[CIY 2 ‘10];[CIY 3 ‘11]

$$F(P,Q) = F_{\mathrm{ST1}}(P,Q) \times F_{\mathrm{ST2}}(P,Q) = 0$$

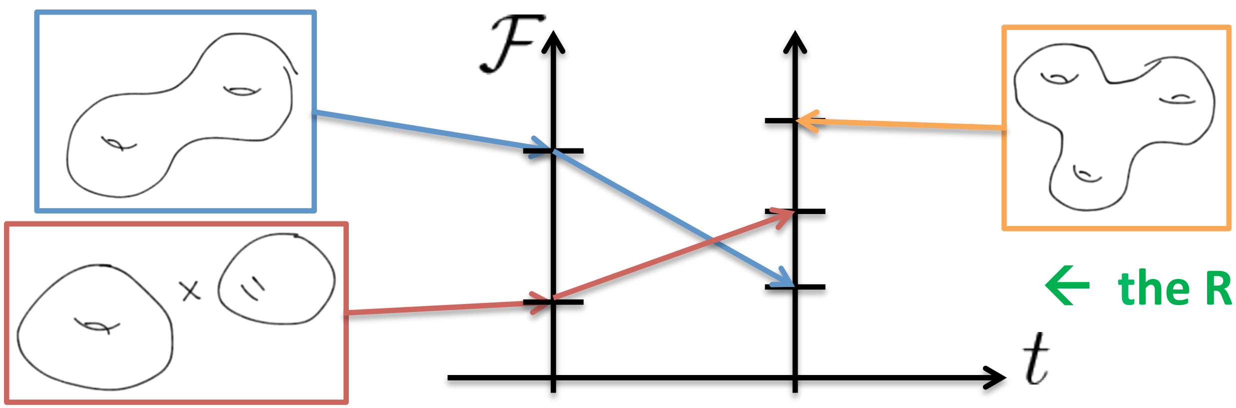




← **Gluing the spectral curves (STs) Non-perturbatively (Today’s topic)**

2) Different perturbative string-theory vacua in the landscape:

[CISY ‘09]; [CIY 2 ‘10]



← **the Riemann-Hilbert problem ([FIKN] for PII, 2-cut)**

We can study the string-theory landscape from the first principle!

2. Stokes phenomenon and isomonodromy systems

Ref) Stokes phenomena and isomonodromy systems
[Moore ‘91] [FIKN‘06] [CIY 2 ‘10]

The ODE systems for determinant operators (FZZT-branes)

$$W(x) = \left\langle \frac{1}{N} \mathrm{tr} \frac{1}{x - M} \right\rangle \Rightarrow \psi(x) = \langle \det(x - X) \rangle = \langle e^{\ln(x-X)} \rangle$$

$g \to 0 \ (\Leftrightarrow N \to \infty)$

$$\psi(x) \sim e^{\frac{1}{g}\varphi(x)}, \quad \text{fixed } {}^{\exists}x$$

The resolvent, i.e. the spectral curve: $W(x) = \partial_x \varphi(x)$

Generally, this satisfies the following kind of linear ODE systems:

$$\boxed{g \frac{\partial}{\partial x} \psi(x) = \mathcal{Q}(x) \psi(x)}$$

k-cut ⇔ k x k matrix Q
[Fukuma-HI '06];[CIY 2 '10]

For simplicity, we here assume:

Poincaré index r

$$\mathcal{Q}(x) = \mathcal{Q}_0 x^{r-1} + \mathcal{Q}_1 x^{r-2} + \cdots + \mathcal{Q}_{r-1}$$

$$\mathcal{Q}_0 = \Omega^{-\gamma}, \quad \Omega = \mathrm{diag}(1, \omega, \omega^2, \cdots, \omega^{k-1})$$

Stokes phenomenon of Airy function

Airy function: $\left(\frac{d^2}{d\zeta^2}-\zeta\right)\psi(\zeta)=0 \qquad \psi(\zeta)=Ai(\zeta),\, Bi(\zeta)$

$$Bi(\zeta)=e^{\frac{\pi}{6}i}Ai(e^{\frac{2}{3}\pi i}\zeta)+e^{-\frac{\pi}{6}i}Ai(e^{-\frac{2}{3}\pi i}\zeta)$$

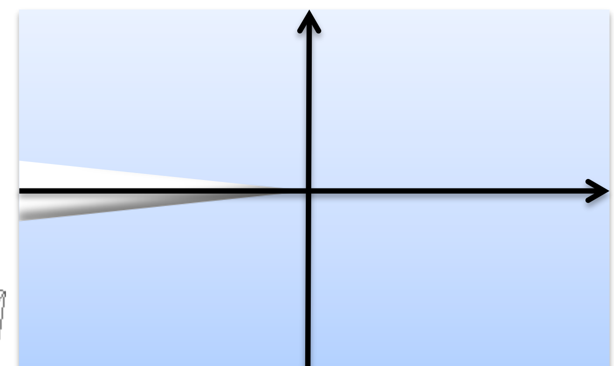
$\boxed{\zeta\to+\infty}$

$$Ai(\zeta)\simeq\frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{2\pi}\left(\frac{3}{4}\right)^n\boxed{\frac{\Gamma(n+\frac{1}{6})\Gamma(n+\frac{5}{6})}{n!}}\zeta^{-\frac{3}{2}n}+\mathcal{O}(e^{-*\zeta^*})\right]$$

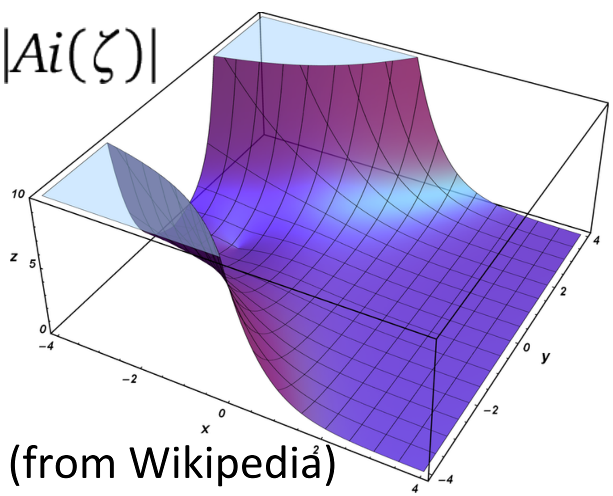
$\sim n!$

Asymptotic expansion!

This expansion is valid in $\zeta\to\infty,\ |\arg(\zeta)|<\pi$

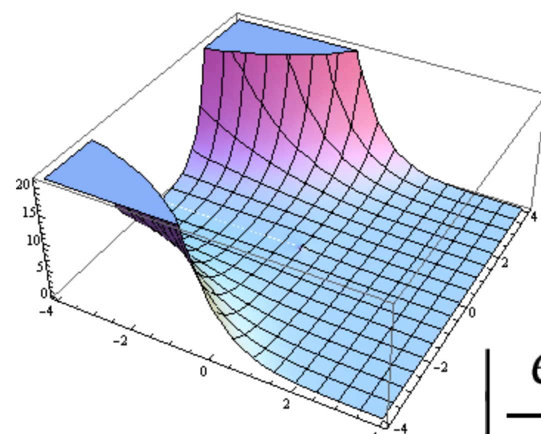


$|Ai(\zeta)|$



(from Wikipedia)

$\approx$



$$\left|\frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\right|$$

Stokes phenomenon of Airy function

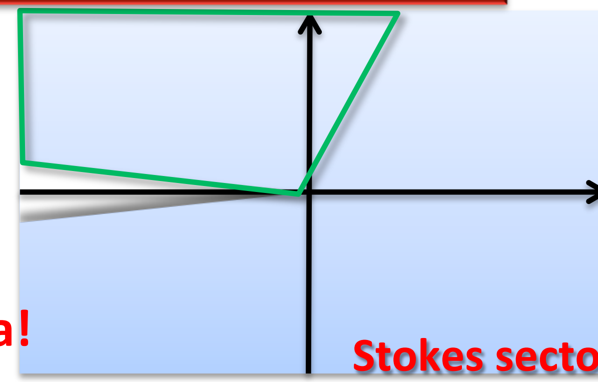
1. Asymptotic expansions are only applied in specific angular domains (**Stokes sectors**)
2. Differences of the expansions in the intersections are only by **relatively and exponentially small terms**

Airy function:

$$\zeta \to +\infty$$

$$Ai(\zeta) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1+\cdots\right]$$

(valid in $\zeta \to \infty, \ |\arg(\zeta)| < \pi$)



Stokes Data!

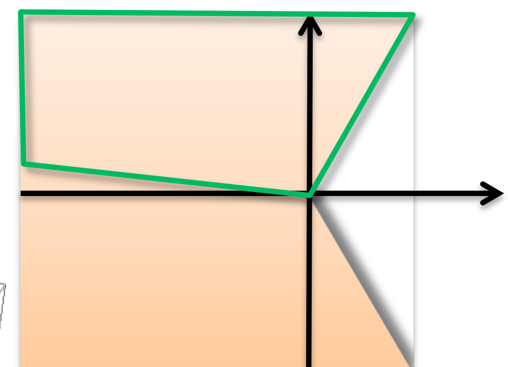
Stokes multiplier

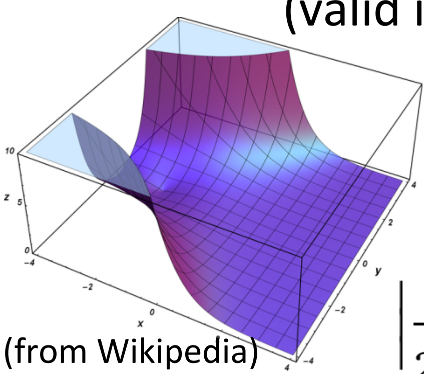
(relatively) Exponentially small !

$$\zeta \to \infty \times e^{\pi i}$$

$$Ai(\zeta) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1+\cdots\right] + i\frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1+\cdots\right]$$

(valid in $\zeta \to \infty, \ |\pi - \arg(\zeta)| < \frac{2\pi}{3}$)





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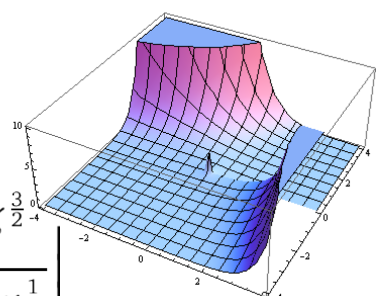
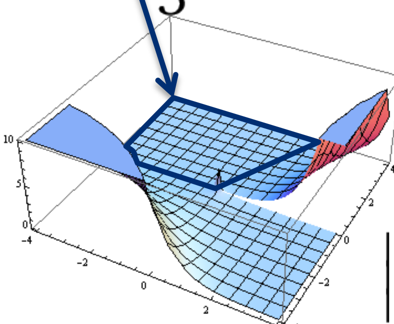
$\approx$  $+$ 

$$\left|\frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\right| \qquad \left|\frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\right|$$

Stokes phenomenon of Airy function

Airy function: $\left(\frac{d^2}{d\zeta^2}-\zeta\right)\psi(\zeta)=0 \qquad \psi(\zeta)=Ai(\zeta),\ Bi(\zeta)$

$$\left(Bi(\zeta)=e^{\frac{\pi}{6}i}Ai(e^{\frac{2}{3}\pi i}\zeta)+e^{-\frac{\pi}{6}i}Ai(e^{-\frac{2}{3}\pi i}\zeta)\right)$$

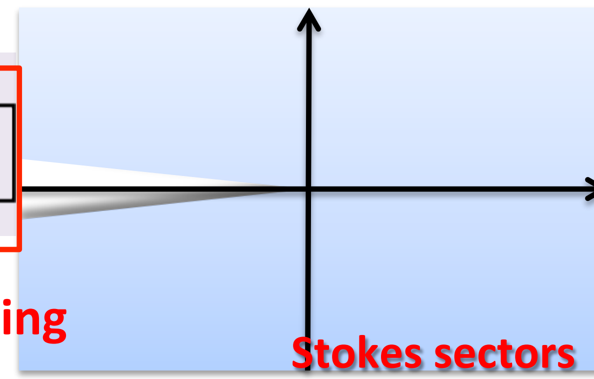
$\boxed{\zeta\to+\infty}$

$$Ai(\zeta) \qquad \simeq \boxed{\frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1+\cdots\right]}$$

(valid in $\zeta\to\infty,\ |\arg(\zeta)|<\pi$)

different

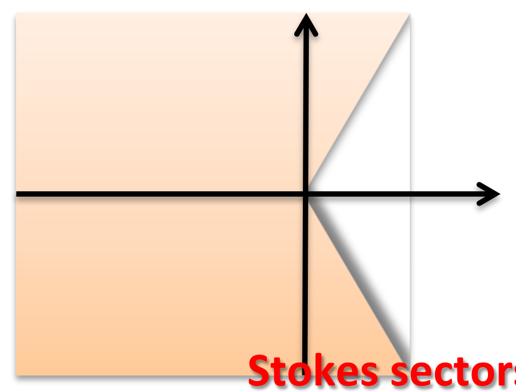
Keep using



$\boxed{\zeta\to\infty\times e^{\pi i}}$

$$Ai(\zeta)-i\left(e^{-\frac{\pi}{6}i}Ai(e^{-\frac{2}{3}\pi i}\zeta)\right)\simeq\boxed{\frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1+\cdots\right]}$$

(valid in $\zeta\to\infty,\ |\pi-\arg(\zeta)|<\frac{2\pi}{3}$)



Stokes phenomenon of the ODE of the matrix models

$$\boxed{g\frac{\partial}{\partial x}\psi(x) = \mathcal{Q}(x)\psi(x)}$$

$$\left(\begin{aligned} &\mathcal{Q}(x) = \mathcal{Q}_0 x^{r-1} + \mathcal{Q}_1 x^{r-2} + \cdots + \mathcal{Q}_{r-1} \\ &\mathcal{Q}_0 = \Omega^{-\gamma}, \quad \Omega = \mathrm{diag}(1, \omega, \omega^2, \cdots, \omega^{k-1}) \end{aligned}\right)$$

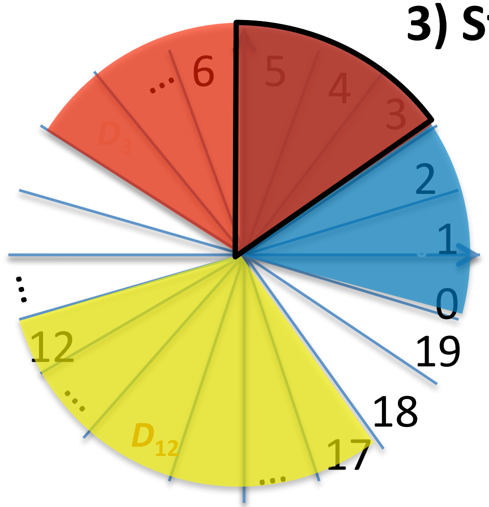
1) ***Complete basis of the asymptotic solutions:***

$$\psi(x) \simeq \psi^{(j)}_{\mathrm{asym}}(x) = \chi^{(j)}(x)\, e^{\frac{1}{g}\varphi^{(j)}(x)} \boxed{\left[1 + O(g)\right]}$$

$$(j = 1, 2, \cdots, k)$$

In the following, we skip this

2) ***Stokes sectors***



3) Stokes phenomena

(relatively and exponentially small terms)

Stokes phenomenon of the ODE of the matrix models

$$g\frac{\partial}{\partial x}\psi(x)=\mathscr{Q}(x)\psi(x)$$

$$\left(\begin{aligned}&\mathscr{Q}(x)=\mathscr{Q}_0x^{r-1}+\mathscr{Q}_1x^{r-2}+\cdots+\mathscr{Q}_{r-1}\\ &\mathscr{Q}_0=\Omega^{-\gamma},\quad \Omega=\mathrm{diag}(1,\omega,\omega^2,\cdots,\omega^{k-1})\end{aligned}\right)$$

1) ***Complete basis of the asymptotic solutions:***

Spectral curve

⇔ Perturb. String Theory

$$\psi(x)\simeq\psi^{(j)}_{\rm asym}(x)=\chi^{(j)}(x)\,e^{\frac{1}{g}\varphi^{(j)}(x)}$$

$$(j=1,2,\cdots,k)$$

$$\varphi^{(j)}(x)=\omega^{-\gamma(j-1)}\frac{x^r}{r}+\cdots$$

Here it is convenient to introduce

$$\Psi_{\rm asym}(x)\equiv\left(\psi^{(1)}_{\rm asym}(x),\cdots,\psi^{(k)}_{\rm asym}(x)\right)$$

$$\left(=\left(\chi^{(1)}(x),\cdots,\chi^{(k)}(x)\right)\boxed{\exp\left[\frac{1}{g}\varphi(x)\right]}\right)$$

$$\begin{pmatrix}e^{\frac{1}{g}\varphi^{(1)}(x)} & & \\ & \ddots & \\ & & e^{\frac{1}{g}\varphi^{(k)}(x)}\end{pmatrix}$$

General solutions:

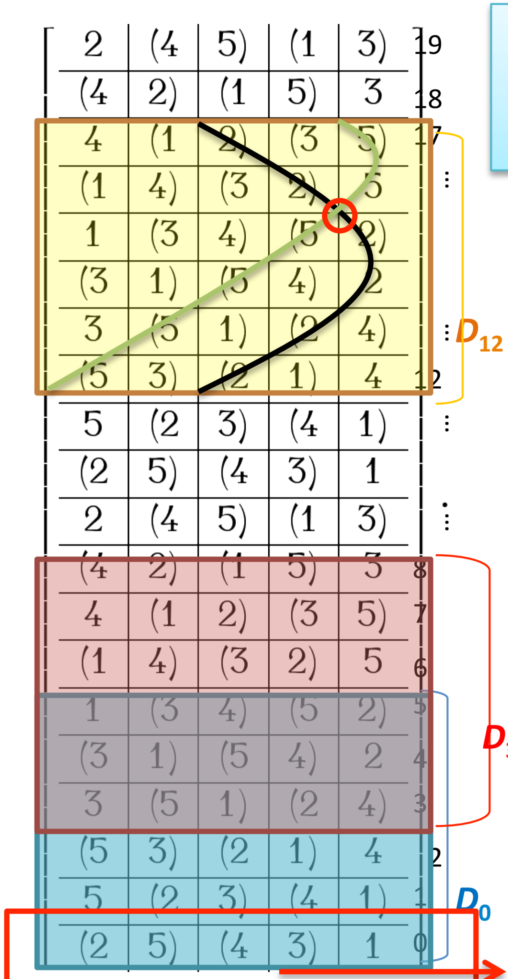
$$\psi_{\vec{C}}(x)\simeq\Psi_{\rm asym}(x)\vec{C}$$

Superposition of wavefunction with different perturbative string theories

Stokes phenomenon of the ODE of the matrix models

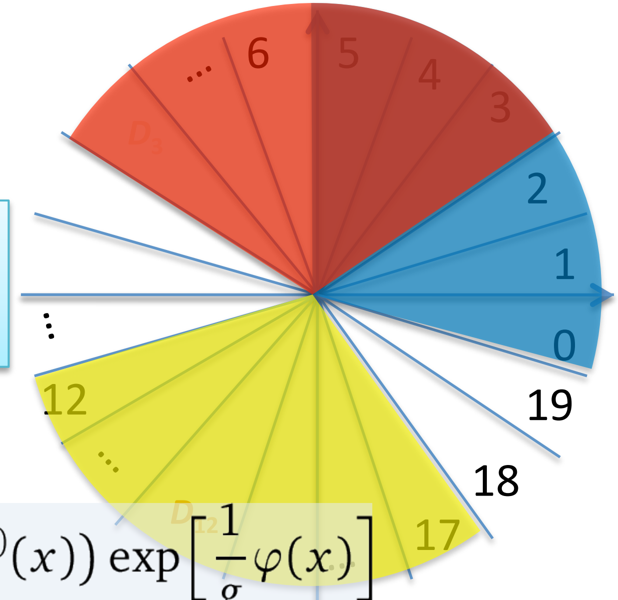
2) *Stokes sectors, and Stokes matrices*

E.g.) r=2, 5 x 5, γ=2 (Z_5 symmetric)



Stokes sectors

$$D_n \ (n = 0, 1, \cdots, 2rk-1)$$



$$\Psi_{12}(x) \simeq \boxed{\Psi_{\rm asym}(x)}$$

$$\Psi_{\rm asym}(x) = (\chi^{(1)}(x), \cdots, \chi^{(k)}(x)) \exp\left[\frac{1}{g}\varphi(x)\right]$$

Canonical solutions (exact solutions)

$$\Psi_n(x) \simeq \boxed{\Psi_{\rm asym}(x)} \, (x \to \infty \in D_n)$$

Keep using

$$\Psi_3(x) \simeq \boxed{\Psi_{\rm asym}(x)}$$

$$\Psi_0(x) \simeq \boxed{\Psi_{\rm asym}(x)}$$

Stokes matrices

$$\Psi_{n+1}(x) = \Psi_n(x) S_n$$

How change the dominance

$$\boxed{|e^{\varphi^{(2)}}| < |e^{\varphi^{(5)}}| < |e^{\varphi^{(4)}}| < |e^{\varphi^{(3)}}| < |e^{\varphi^{(1)}}|}$$

Stokes phenomenon of the ODE of the matrix models

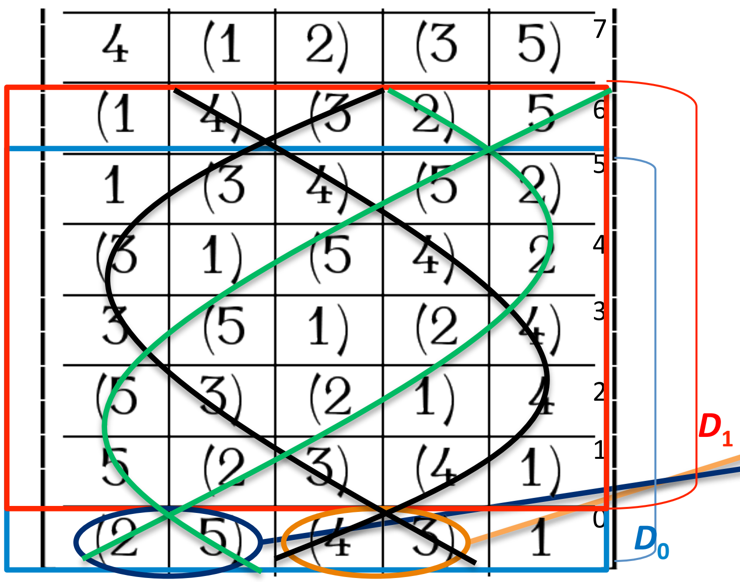
3) *How to read the Stokes matrices?* :Prifile of exponents [CIY 2 '10]

E.g.) r=2, 5 x 5, γ=2 (Z_5 symmetric)

$$\Psi_1(x) = \Psi_0(x) S_0$$

Stokes matrices

$$\Psi_{n+1}(x) = \Psi_n(x) S_n$$



$$S_0 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & s_{0,3,4} & \\ & & & 1 & \\ & s_{0,5,2} & & & 1 \end{pmatrix}$$

Set of Stokes multipliers !

Thm [CIY2 '10]

$^{\exists}(i|j)_l \Leftrightarrow s_{l,j,i}$: non-trivial

Inverse monodromy (Riemann-Hilbert) problem [FIKN]

Direct monodromy problem

$$g\frac{\partial}{\partial x}\psi(x)=\mathcal{Q}(x)\psi(x)$$

Given $\mathcal{Q}(x)$

WKB

Solve $\{\Psi_n(x)\}_{n=0}^{2rk-1}$

Obtain

$$S_n=\Psi_{n+1}(x)\Psi_n(x)^{-1}$$

Analytic problem

Inverse monodromy problem

Consistency (Algebraic problem)

Given: **Stokes matrices**

$$\Psi_{n+1}(x)=\Psi_n(x)S_n$$

Special Stokes multipliers which satisfy physical constraints

$$\Psi_{\mathrm{asym}}(x)=\mathscr{X}(x)\exp\left[\frac{1}{g}\varphi(x)\right]$$

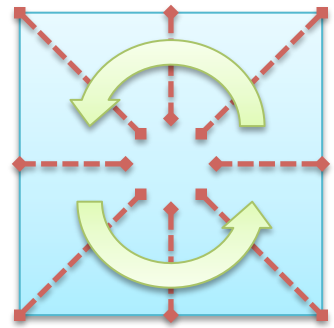
Obtain

$$\mathcal{Q}(x)=g\frac{\partial}{\partial x}\Psi_{\mathrm{asym}}(x)\cdot\Psi_{\mathrm{asym}}(x)^{-1}$$

Algebraic relations of the Stokes matrices

1. Z_k –symmetry condition

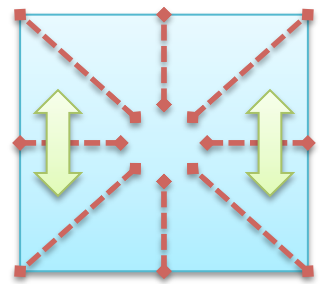
$$S_{n+2r} = \Gamma^{-1} S_n \Gamma$$



$$\Gamma = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}$$

2. Hermiticity condition

$$S_n^* = \Delta \Gamma S_{(2r-1)k-n}^{-1} \Delta \Gamma$$



$$\Delta = \begin{pmatrix} & & 1 \\ & \cdots & \\ 1 & & \\ 1 & & \end{pmatrix}$$

3. Monodromy Free condition

most difficult part!

$$\boxed{S_0 S_1 S_2 \cdots S_{2rk-1} = I_k}$$

$$\left(\Psi_n(e^{2\pi i} x) = \Psi_n(x) \right)$$

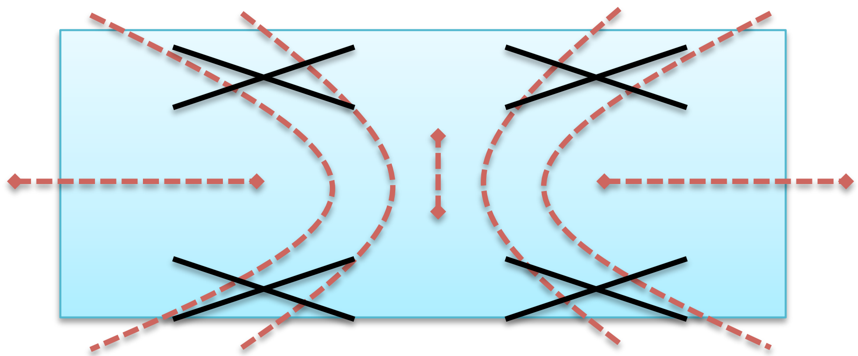
4. Physical constraint: **The multi-cut boundary condition**

This helps us to obtain solutions for general (k,r)

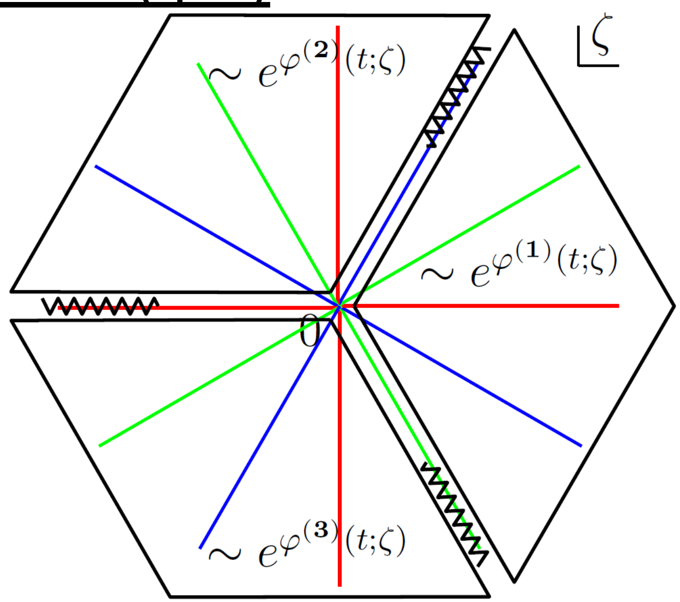
3. Stokes phenomenon in non-critical string theory

Ref) Stokes phenomena and quantum integrability
[CIY2 ‘10][CIY3 ‘11]

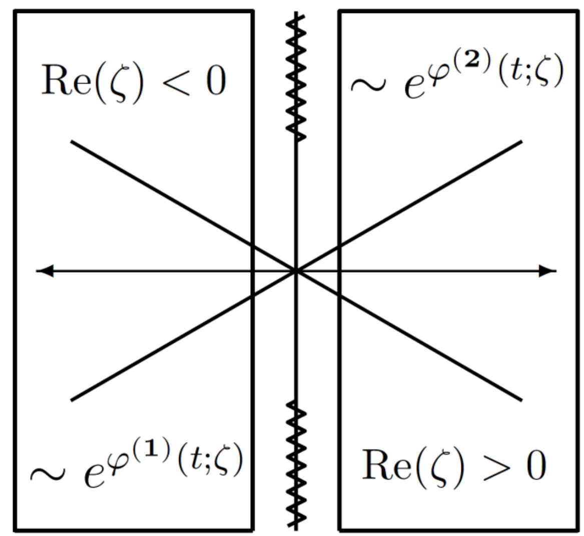
Multi-cut boundary condition



<u>3-cut case (q=1)</u>



<u>2-cut case (q=2: pureSUGRA)</u>



Stokes phenomenon of Airy function

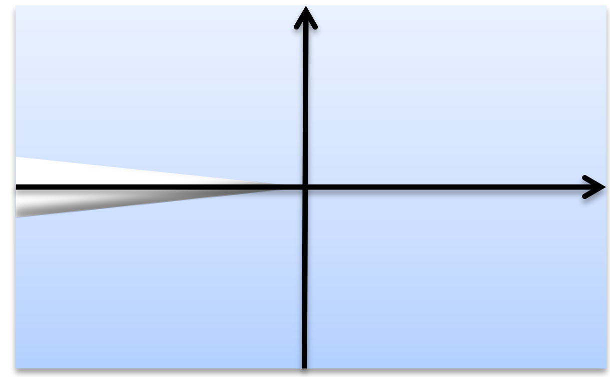
Airy function: $\left(\frac{d^2}{d\zeta^2} - \zeta\right)\psi(\zeta) = 0 \qquad \psi(\zeta) = Ai(\zeta),\, Bi(\zeta)$

$$Bi(\zeta) = e^{\frac{\pi}{6}i} Ai(e^{\frac{2}{3}\pi i}\zeta) + e^{-\frac{\pi}{6}i} Ai(e^{-\frac{2}{3}\pi i}\zeta)$$

$\boxed{\zeta \to +\infty}$

$$Ai(\zeta) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1 + \cdots\right]$$

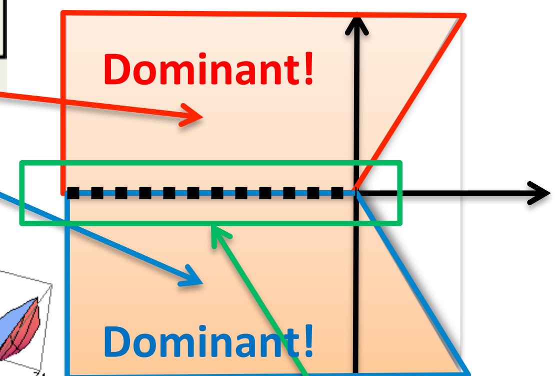
(valid in $\zeta \to \infty,\ |\arg(\zeta)| < \pi$)



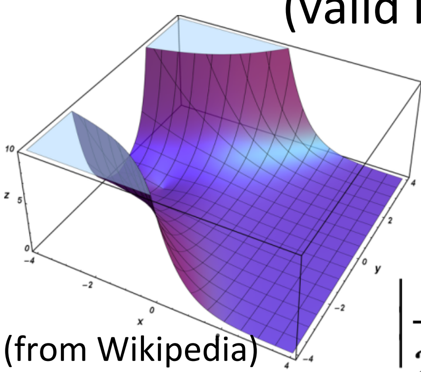
$\boxed{\zeta \to \infty \times e^{\pi i}}$

$$Ai(\zeta) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1 + \cdots\right] + i\frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1 + \cdots\right]$$

(valid in $\zeta \to \infty,\ |\pi - \arg(\zeta)| < \frac{2\pi}{3}$)

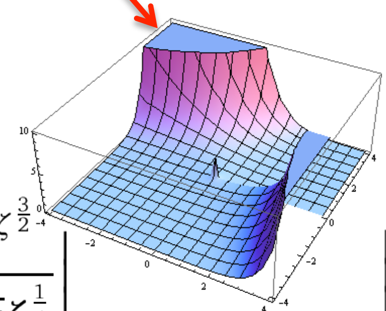


Change of dominance
(Stokes line)

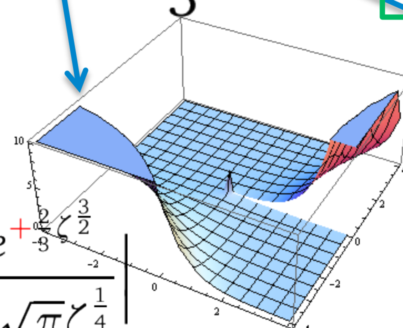


(from Wikipedia)

$\approx \left|\frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\right|$



$+ \left|\frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\right|$



Stokes phenomenon of Airy function

Airy system ⇔ (2,1) topological minimal string theory

$$Ai(x) = \langle \det(x - X) \rangle \sim e^{\frac{1}{g}\varphi(x)} \qquad W(x) = \partial_x \varphi(x)$$

$$\boxed{W(x \pm i\epsilon) = \mp\sqrt{x}}$$ **discontinuity**

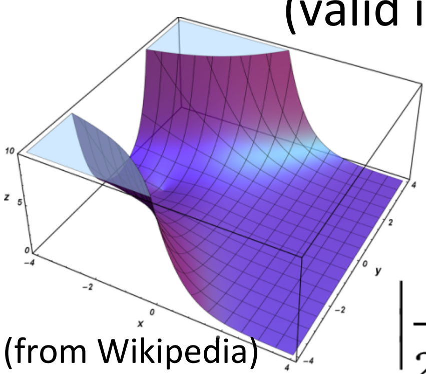
⇔ **Eigenvalue cut of the matrix model**

Physical cuts = lines with dominance change (Stokes lines) [MMSS '05]

$$\boxed{\zeta \to \infty \times e^{\pi i}}$$

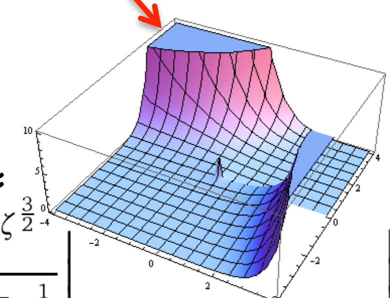
$$Ai(\zeta) \simeq \frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1 + \cdots\right] + i\frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\left[1 + \cdots\right]$$

(valid in $\zeta \to \infty, \ |\pi - \arg(\zeta)| < \frac{2\pi}{3}$)

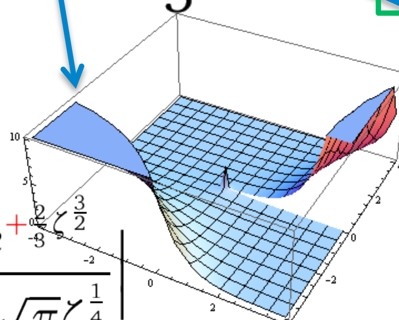


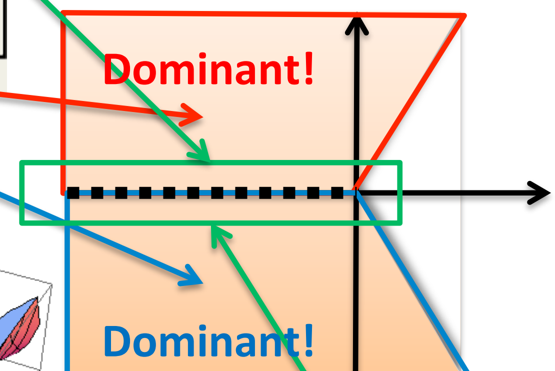
(from Wikipedia)

$$\approx \left|\frac{e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\right|$$



$$+ \left|\frac{e^{+\frac{2}{3}\zeta^{\frac{3}{2}}}}{2\sqrt{\pi}\zeta^{\frac{1}{4}}}\right|$$

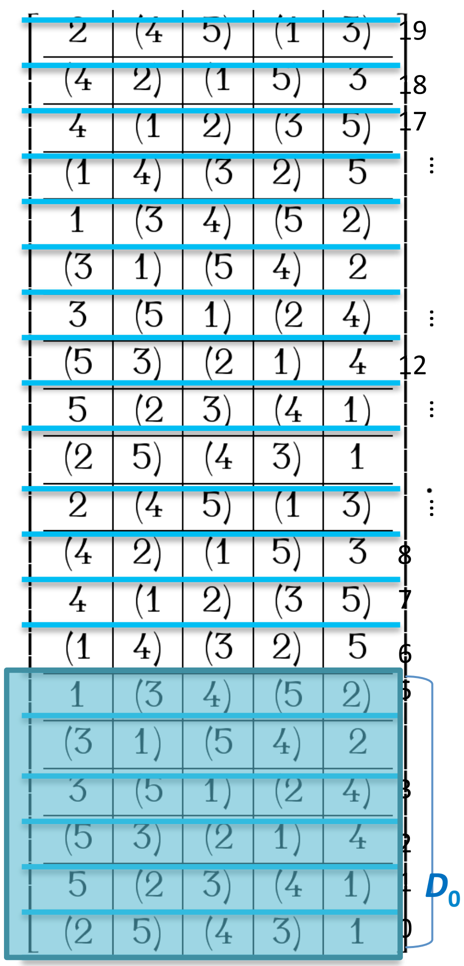


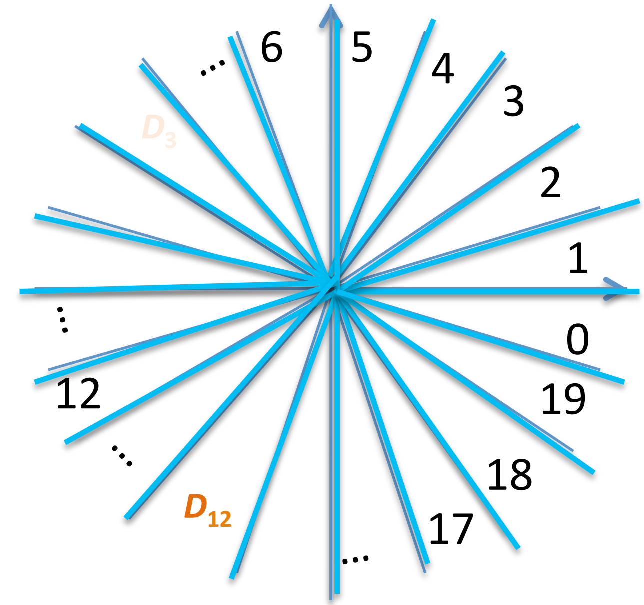


Change of dominance **(Stokes line)**

Multi-cut boundary condition [CIY 2 ‘10]

E.g.) r=2, 5 x 5, γ=2 (Z_5 symmetric)





$$\psi_C(x) = \Psi_0 C^{(0)} = \Psi_1 C^{(1)} = \cdots = \Psi_{2rk} C^{(2rk)}$$
$$\simeq \psi^{(1)}_{\rm asy} c^{(n)}_1 + \psi^{(2)}_{\rm asy} c^{(n)}_2 + \psi^{(3)}_{\rm asy} c^{(n)}_3 + \psi^{(4)}_{\rm asy} c^{(n)}_4 + \psi^{(5)}_{\rm asy} c^{(n)}_5$$
$$(x \to \infty \in D_n)$$

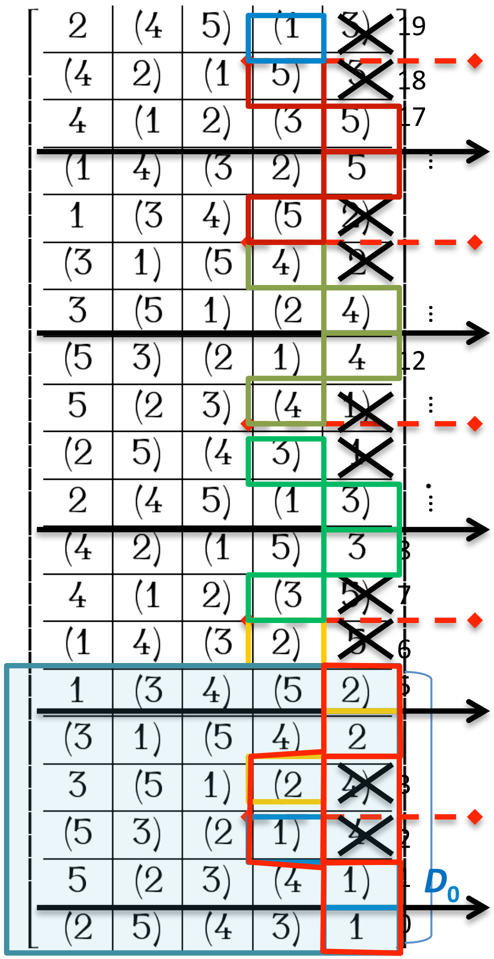
All the horizontal lines are Stokes lines!

All lines are candidates of the cuts!

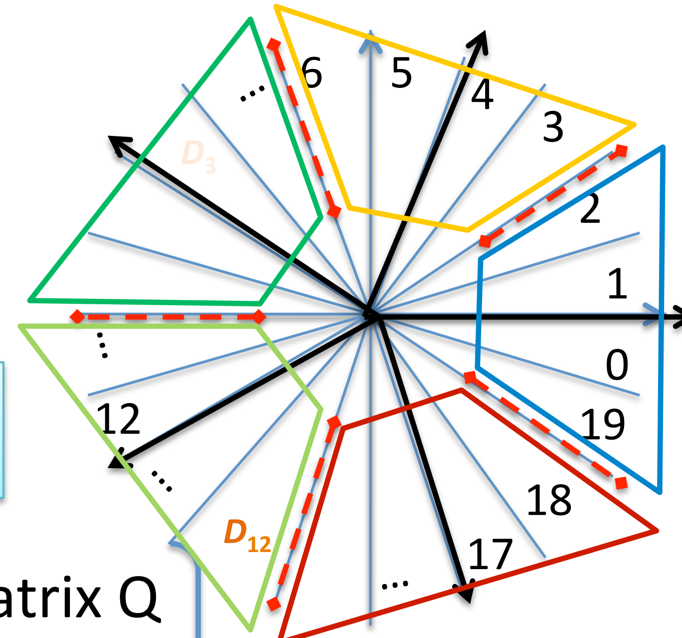
$$\left|e^{\varphi^{(2)}}\right| < \left|e^{\varphi^{(5)}}\right| < \left|e^{\varphi^{(4)}}\right| < \left|e^{\varphi^{(3)}}\right| < \left|e^{\varphi^{(1)}}\right|$$

Multi-cut boundary condition [CIY 2 '10]

E.g.) r=2, 5 x 5, γ=2 (Z_5 symmetric)



We choose "k" of them as physical cuts!



k-cut ⇔ k x k matrix Q

[Fukuma-HI '06];[CIY 2 '10]

$$\psi_C(x) = \Psi_0 C^{(0)} = \Psi_1 C^{(1)} = \cdots = \Psi_{2rk} C^{(2rk)}$$

$$\simeq \boxed{\psi_{\mathrm{asy}}^{(1)} \boxed{c_1^{(0)}}} + \boxed{\psi_{\mathrm{asy}}^{(2)} \boxed{c_2^{(0)}}} + \psi_{\mathrm{asy}}^{(3)} c_3^{(0)} + \boxed{\psi_{\mathrm{asy}}^{(4)} \cancel{c_4^{(0)}}} + \psi_{\mathrm{asy}}^{(5)} c_5^{(0)}$$

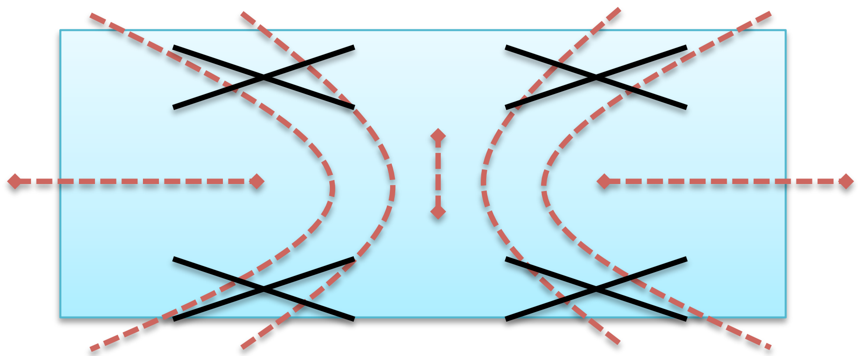
≠0 **≠0** $(x \to \infty \in D_0)$ **=0**

Constraints on Sn

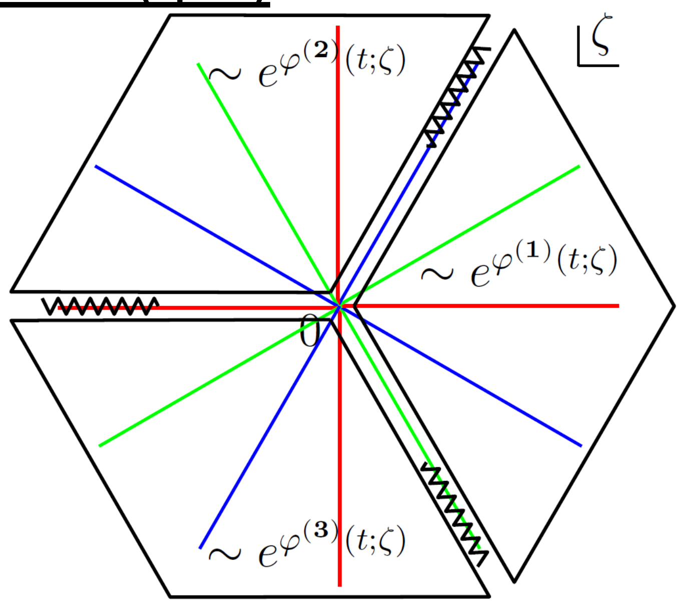
$$\boxed{C^{(n)} = S_n C^{(n+1)}}$$

$$\boxed{|e^{\varphi^{(2)}}| < |e^{\varphi^{(5)}}| < |e^{\varphi^{(4)}}| < |e^{\varphi^{(3)}}| < |e^{\varphi^{(1)}}|}$$

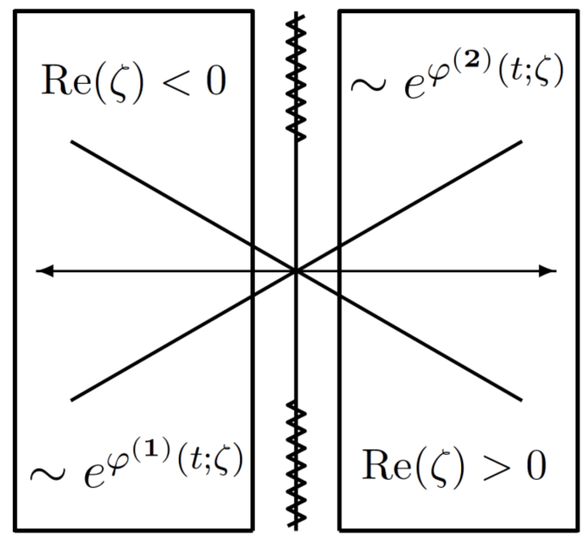
Multi-cut boundary condition



3-cut case (q=1)



2-cut case (q=2: pureSUGRA)

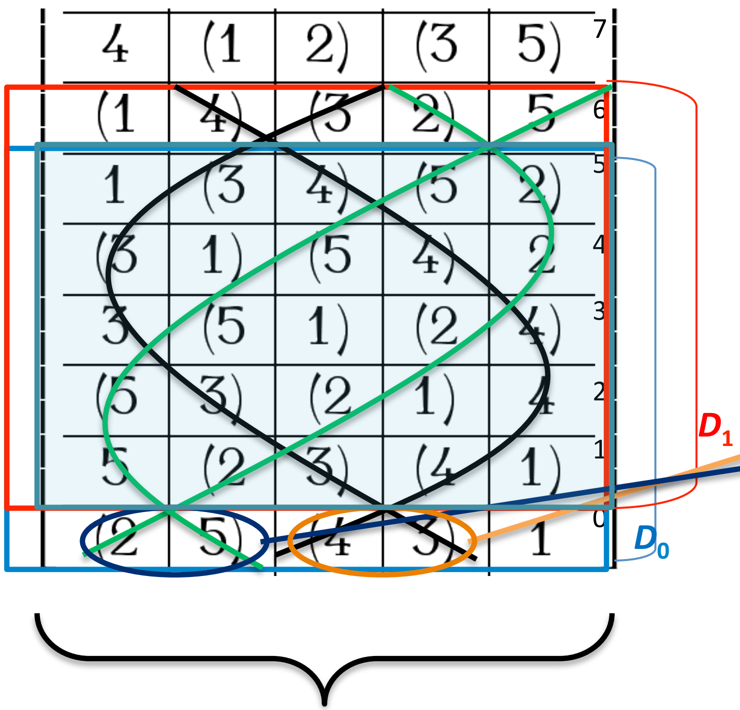


The set of non-trivial Stokes multipliers?

Use Prifile of dominant exponents [CIY 2 ‘10]

E.g.) r=2, 5 x 5, γ=2 (Z_5 symmetric)

$$\Psi_1(x) = \Psi_0(x) S_0$$



Set of Stokes multipliers !

$$S_0 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & s_{0,3,4} & \\ & & & 1 & \\ & s_{0,5,2} & & & 1 \end{pmatrix}$$

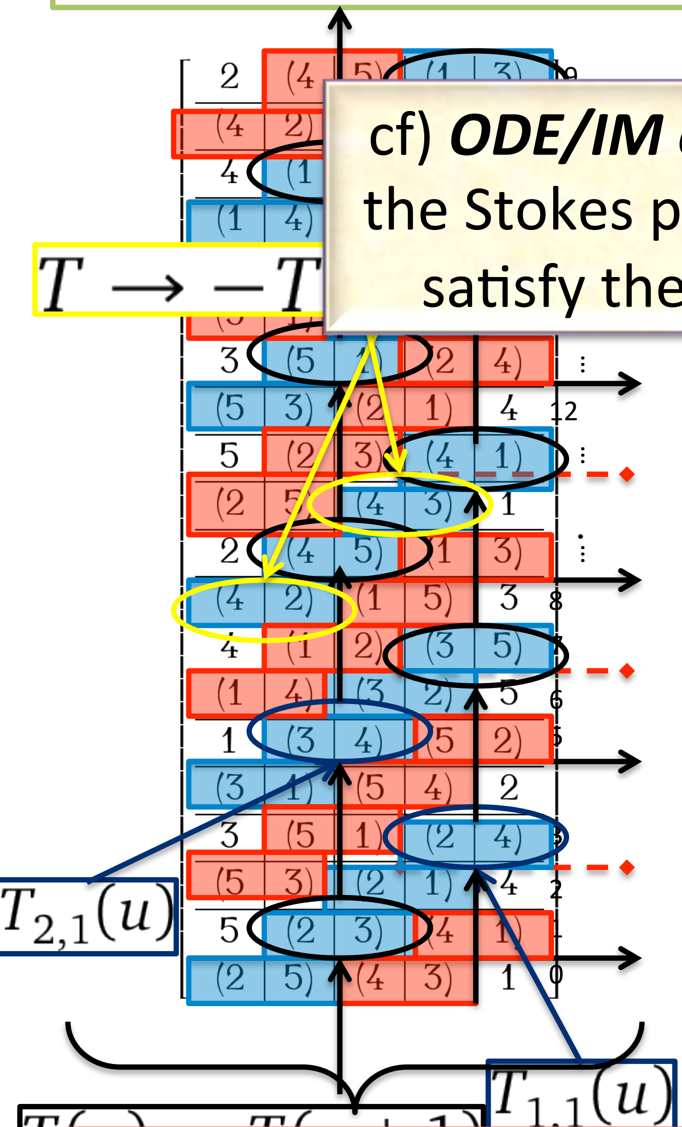
Thm [CIY2 ‘10]

$^{\exists}(i|j)_l \Leftrightarrow s_{l,j,i}$: non-trivial

Quantum integrability [CIY 3 ‘11]

E.g.) r=2, 5 x 5, γ=2 (Z_5 symmetric)

$$C^{(n)} = S_n C^{(n+1)}$$



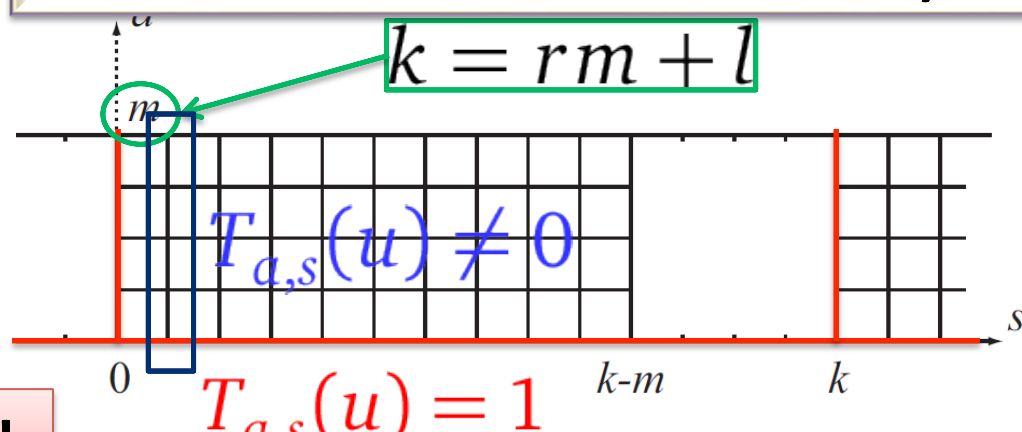
cf) ***ODE/IM correspondence*** [Dorey-Tateo ‘98];[J. Suzuki ‘99]
the Stokes phenomena of special Schrodinger equations satisfy the T-systems of quantum integrable models

Then, the equation becomes T-systems:

$$T_{a,s}(u+1)T_{a,s}(u-1) = T_{a,s+1}(u)T_{a,s-1}(u) + T_{a+1,s}(u)T_{a-1,s}(u)$$

wi

How about the other Stokes multipliers?



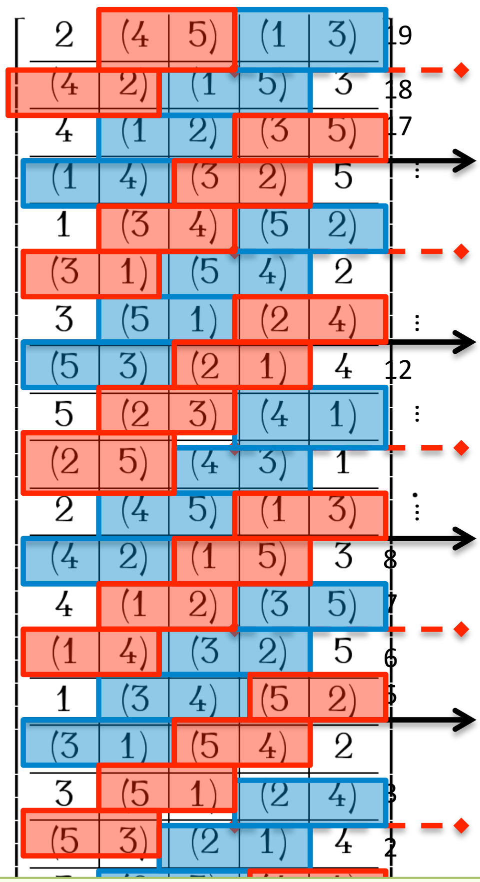
Set of Stokes multipliers !

Complementary Boundary cond. [CIY 3 '11]

Shift the BC !

$$\boxed{\widetilde{C}^{(n)} = S_n \widetilde{C}^{(n+1)}}$$

E.g.) r=2, 5 x 5, γ=2 (Z_5 symmetric)



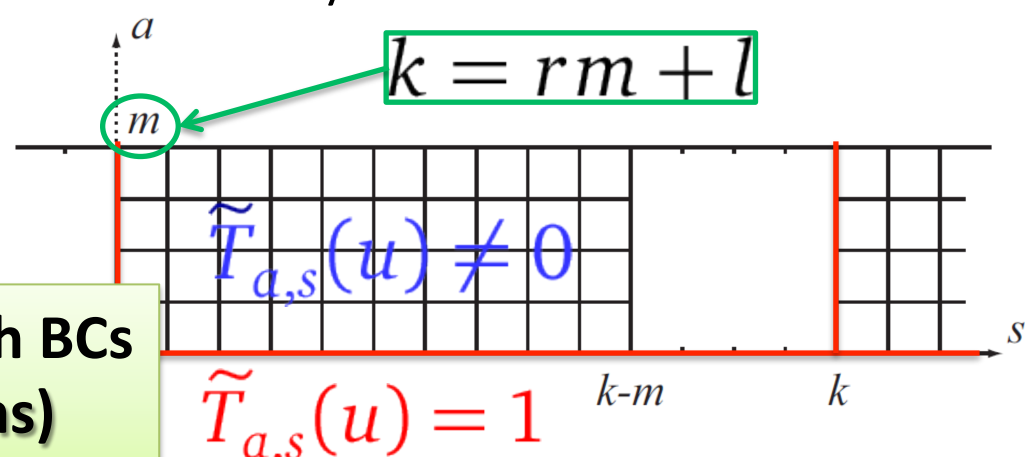
This equation only includes the Stokes multipliers of

$$\boxed{(i|j)_l} \Leftrightarrow s_{l,j,i}$$

Then, the equation becomes T-systems:

$$\widetilde{T}_{a,s}(u+1)\widetilde{T}_{a,s}(u-1) = \widetilde{T}_{a,s+1}(u)\widetilde{T}_{a,s-1}(u) + \widetilde{T}_{a+1,s}(u)\widetilde{T}_{a-1,s}(u)$$

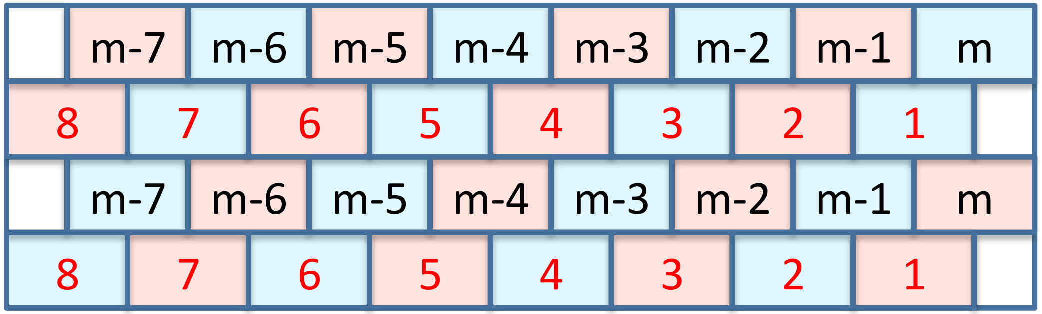
with the boundary condition:



Generally there are "r" such BCs (Coupled multiple T-systems)

Solutions for multi-cut cases

(Ex: r=2, k=2m+1):



$$\left[\begin{array}{c|c|c|c|c} 3 & (5 & 1) & (2 & 4) \\ \hline (5 & 3) & (2 & 1) & 4 \\ \hline 5 & (2 & 3) & (4 & 1) \\ \hline (2 & 5) & (4 & 3) & 1 \end{array}\right]$$

$\boxed{n} = \theta_n^{(0)}$, $\boxed{n} = -(\theta_n^{(0)})^*$, $\boxed{n} = \theta_n^{(2)}$, $\boxed{n} = -(\theta_n^{(2)})^*$

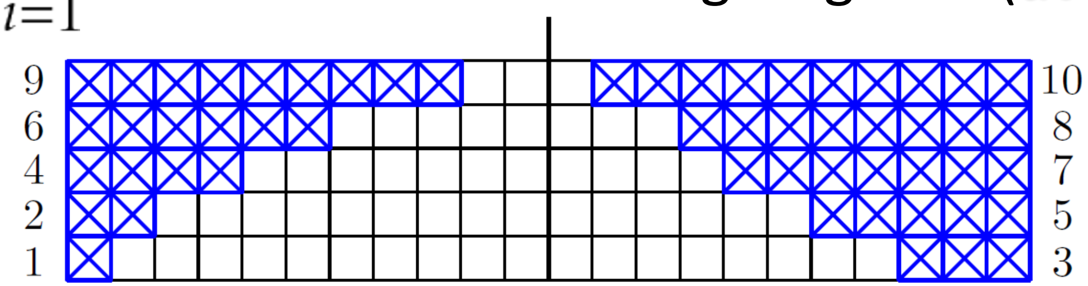
$$\theta_n^{(2p)} = Sym[n, \{\omega^{n_i^{(2p)}}\}_{i=1}^m]$$

[CIY 2 '10] [CIY3 '11]

(Characters of the anti-Symmetric representation of GL)

In addition, they are "coupled multiple T-systems"

$\{n_i^{(0)}; n_i^{(2)}\}_{i=1}^m$ are written with Young diagrams (***avalanches***):



Summary

1. The **D-instanton chemical potentials** are the missing information in the perturbative string theory.
2. This information is responsible for the non-perturbative relationship among perturbative string-theory vacua, and important for study of the string-theory landscape from the first principle.

3. In non-critical string theory, this information is described by **the positions of the physical cuts.**
4. The multi-cut boundary conditions, which turn out to be T-systems of quantum integrable systems, can give a part of the constraints on the non-perturbative system
5. Although physical meaning of the complementary BC is still unclear (in progress [CIY 4 '12]), it allows us to obtain explicit expressions of the Stokes multipliers.

discussions

1. *Physical meaning of the Compl. BCs?*
   → The system is described not only by the resolvent?
   We need other degree of freedom to complete the system?
   (→ FZZT-Cardy branes? [CIY 3 '11]; [CIY4 '12 in progress])

2. D-instanton chemical potentials are determined by "strange constraints" which are expressed as quantum integrability.
   Are there more natural explanations of the multi-cut BC?
   (→ Use Duality? Strong string-coupling description?
   → Non-critical M theory?, Gauge theory?)

Thank you for your attention!