Liouville Quantum Gravity, KPZ & Schramm-Loewner Evolution I

Bertrand Duplantier

Institut de Physique Théorique, Saclay, France

& Scott Sheffield, MIT Math, USA

– UK-Japan Winter School 2012 –

String Theory, Geometry

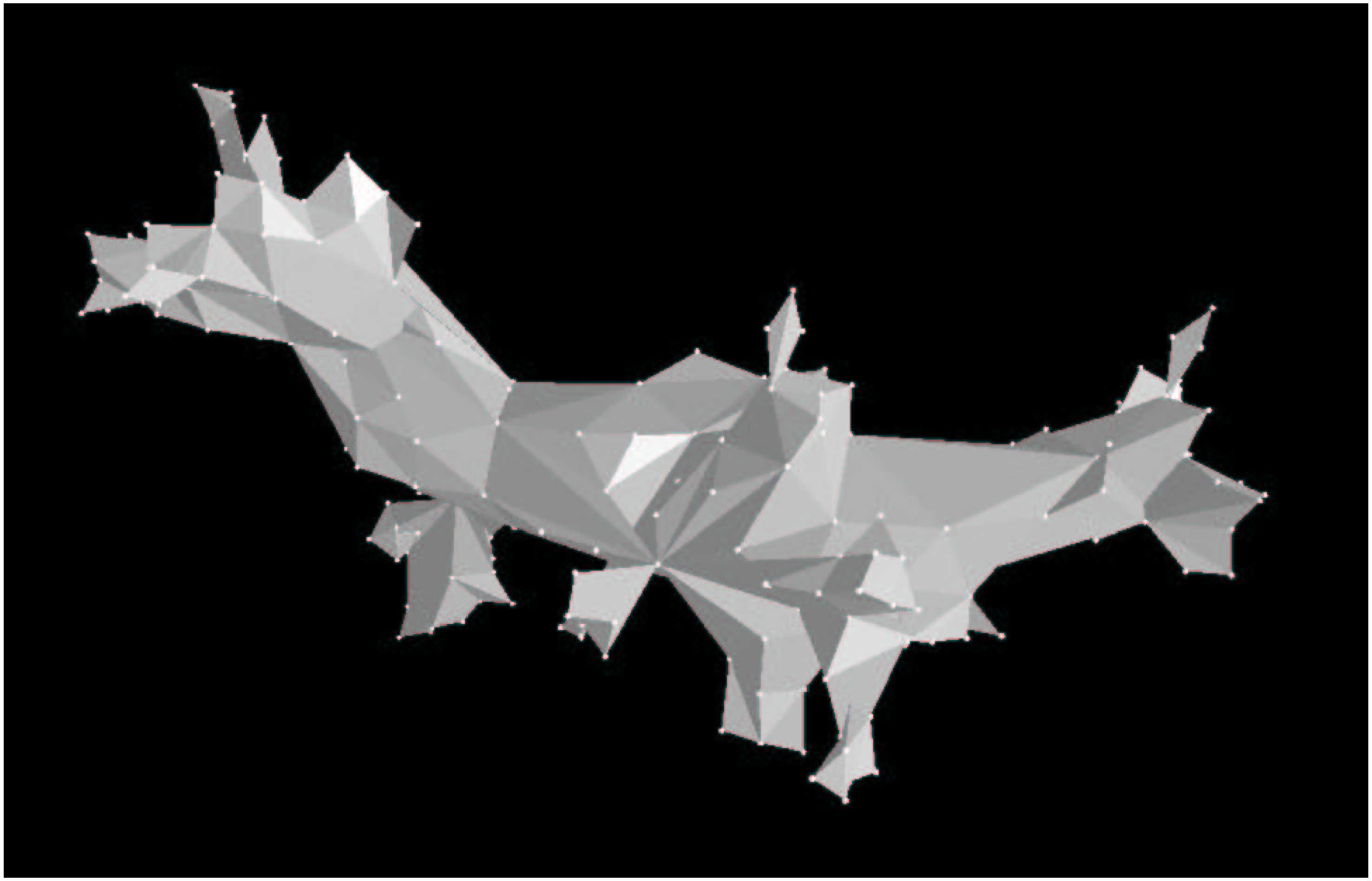
& Mathematical Physics

Department of Physics

University of Oxford / 5 – 8 January 2012

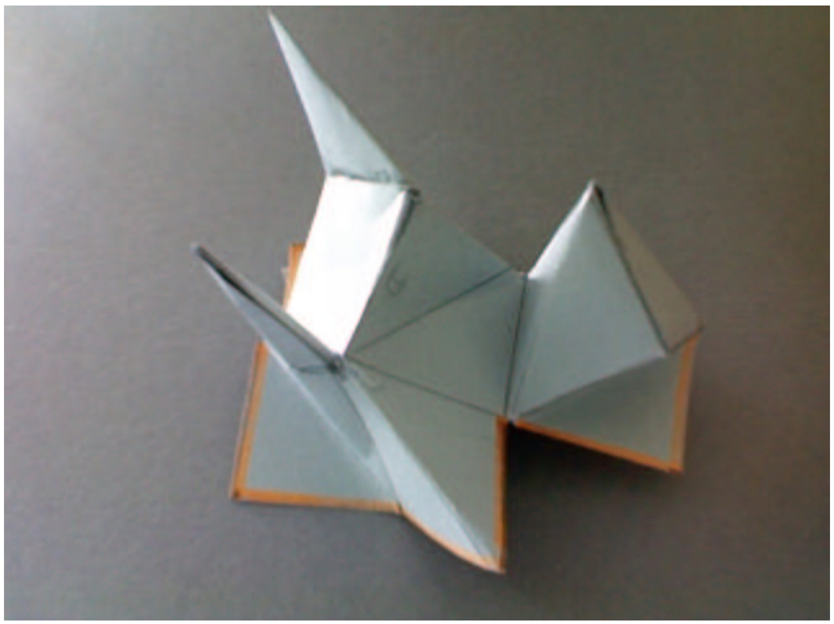
Liouville Quantum Gravity & KPZ

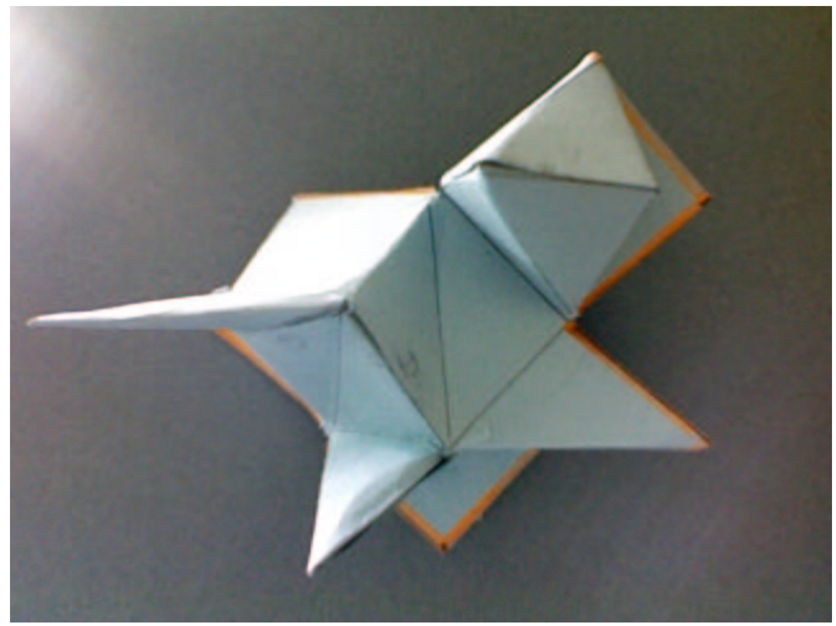
A Random Surface



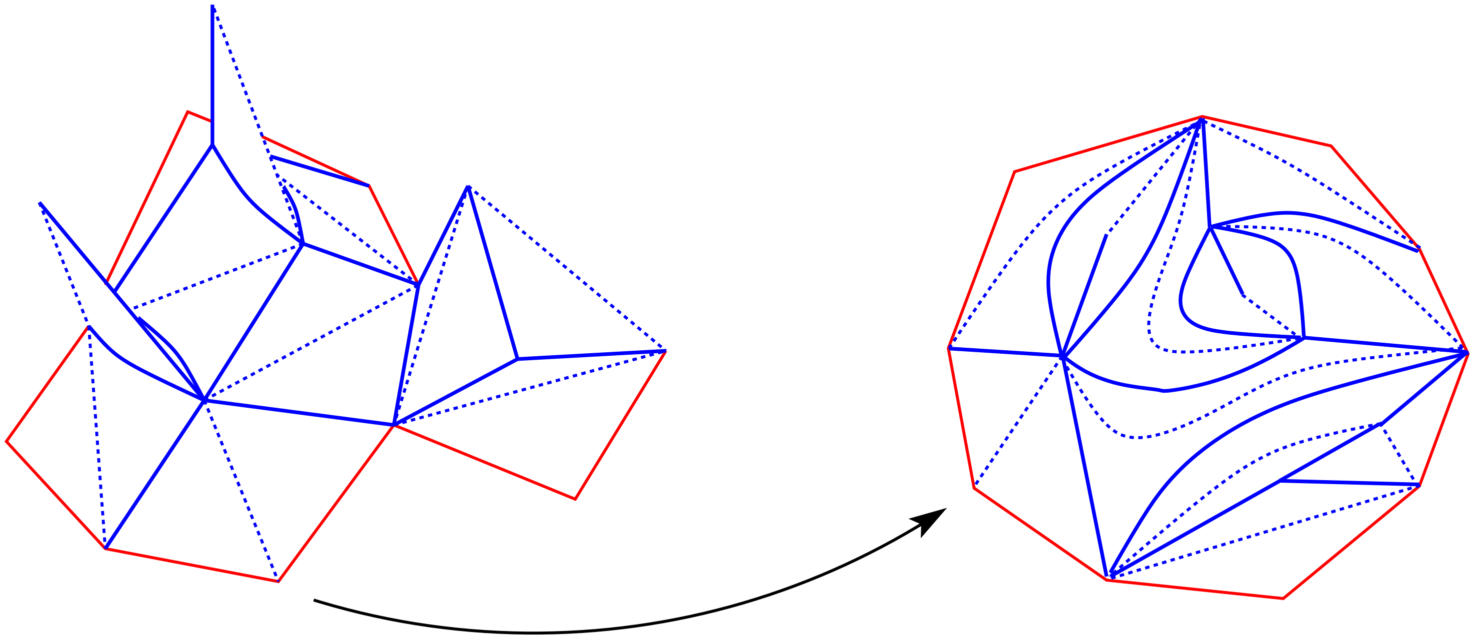
[Courtesy of G. Chapuy (2009)]

A Random Quadrangulation





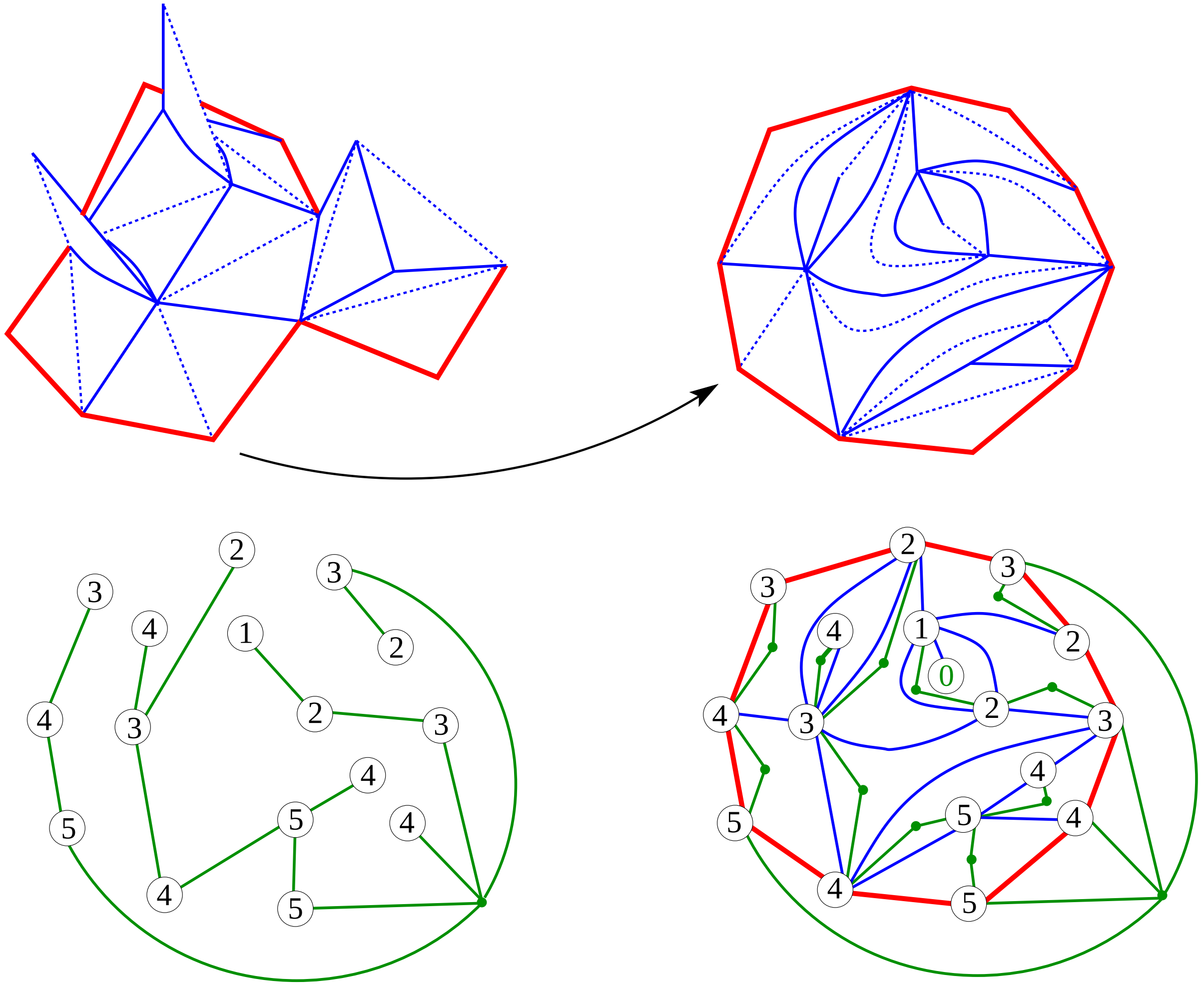
Random Quadrangulation & Random Planar Map



Random Matrices *BIPZ '78; Ambjørn, Durhuus, Fröhlich, Jonsson '83-85; David '85; Boulatov, Kazakov, Kostov, Migdal '85...*

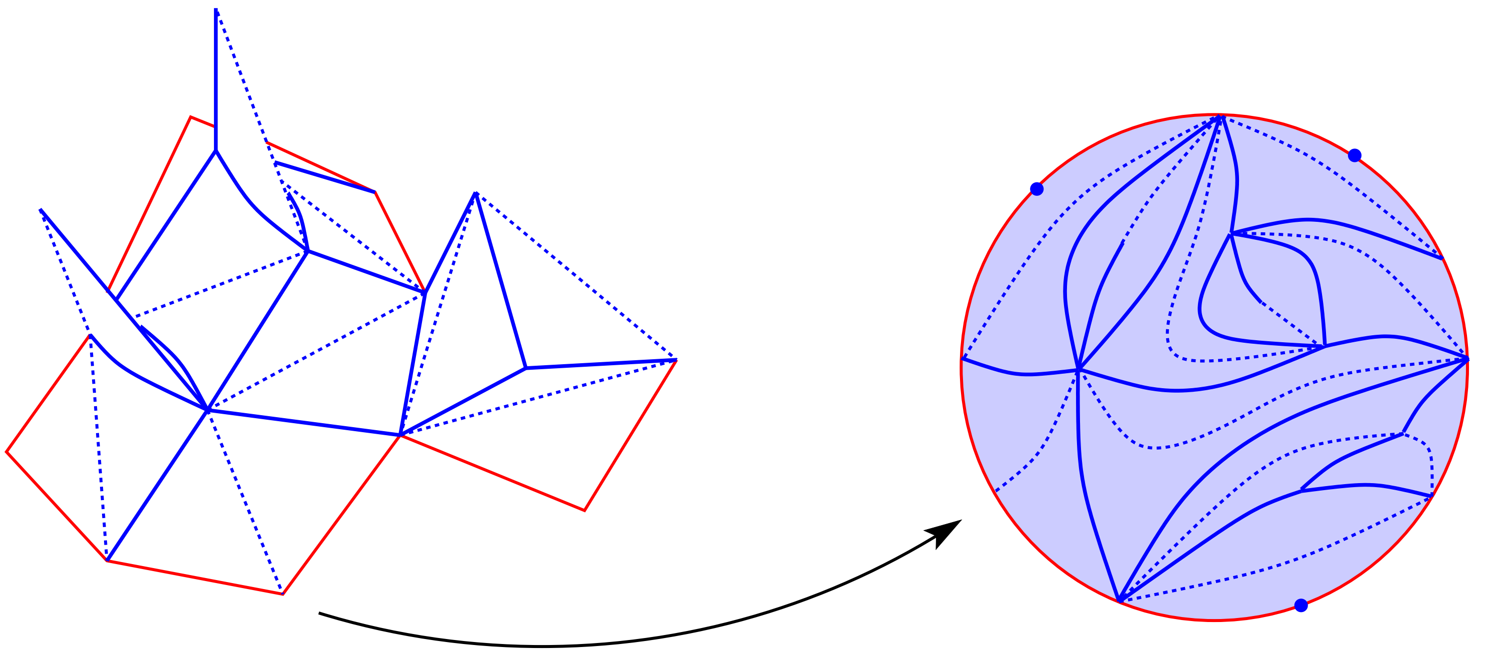
Bijective Combinatorics *Cori, Vauquelin '81; Schaeffer '97; Angel, Schramm '03; Bouttier, Di Francesco, Guitter '04; Le Gall, Miermont...*

Random Quadrangulations & Schaeffer Bijection



Courtesy of E. Guitter (2009)

Random Quadrangulation & Conformal Map to $\mathbb{D}$

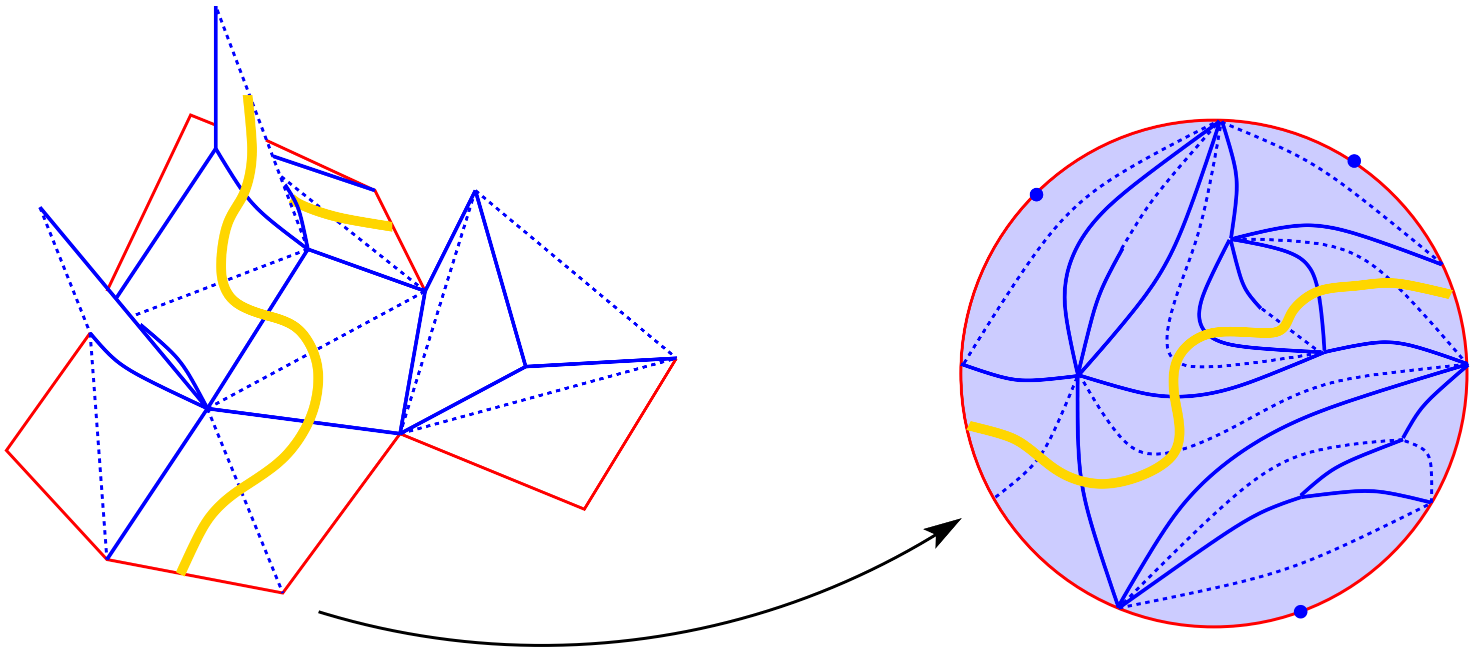


In the continuum scaling limit: **Liouville Quantum Gravity**

A.M. Polyakov '81

Correlation Functions *Seiberg, '90; Goulian, Li '91; Ginsparg, Moore '93; Dorn, Otto '94; Takhtajan '95; Teschner '95; Zamolodchikov^2 '96; Fateev-ZZ '00; Ponsot, Teschner '02; Kostov, Ponsot, Serban '04...*

Random Quadrangulation & Random Sets & Paths



Ising, SAW, $O(N)$ & Potts models: **Random Matrix Models**

Kazakov '86; D. & Kostov '88; Kostov; Daul; Eynard, Zinn-Justin^2...

Bijective Combinatorics *Chassaing & Schaeffer '02;*

Bousquet-Mélou & Schaeffer '02; BDFG '02; Bernardi & B.-M. '09...

Continuum: **Liouville Gravity & Conformal Field Theory**

Thirty Years Ago—*“There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned (and extremely useful) sums over random paths. The replacement is necessary, because today gauge invariance plays the central role in physics. Elementary excitations in gauge theories are formed by the flux lines (closed in the absence of charges) and the time development of these lines forms the world surfaces. All transition amplitudes are given by the sums over all possible surfaces with fixed boundary.”*

A.M. POLYAKOV, *Quantum geometry of bosonic strings*, *Phys. Lett.* **B** 103(3), 207–210 (1981).

Liouville Field Theory (POLYAKOV '81)

$$S_{\mathcal{L}} = \frac{1}{4\pi}\int d^2z\sqrt{\hat{g}}\left(\hat{g}^{ab}\partial_a h\partial_b h + Q\hat{R}h + \lambda e^{\gamma h}\right) [+\mathrm{CFT}]$$

Background metric $\hat{g}$ & curvature $\hat{R}$

Quantum random metric: $g_{ab} = e^{\gamma h}\hat{g}_{ab}$

Quantum area: $\mathcal{A} = \int d^2z\sqrt{\hat{g}}\,e^{\gamma h}$

Conformal invariance & CFT central charge c

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2} = \sqrt{\frac{25-c}{6}}, \quad c \leqslant 1$$

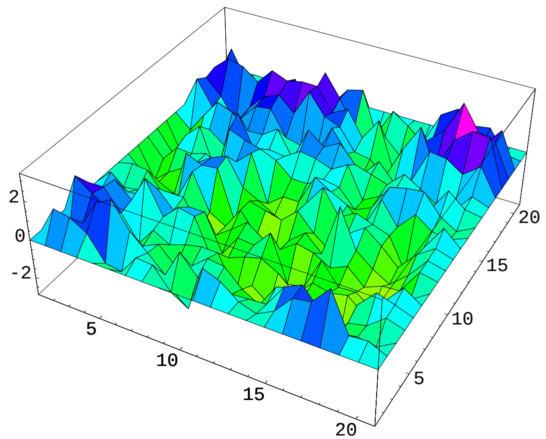
$$\gamma = \frac{1}{\sqrt{6}}\left(\sqrt{25-c} - \sqrt{1-c}\right) = \sqrt{\kappa \wedge 16/\kappa} \leqslant 2 \ (\mathrm{SLE}_\kappa)$$

Gaussian Free Field

In order to separate out what is quite simple from what is complex, and to arrange these matters methodically, we ought, in the case of every series in which we have deduced certain facts the one from the other, to notice which fact is simple, and to mark the interval, greater, less, or equal, which separates all the others from this.

René Descartes, Rules for the Direction of the Mind, VI (1628-1629).

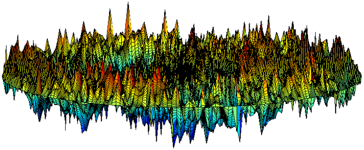
Gaussian Free Field (GFF)



Distribution h with *Gaussian weight* $\exp\left[-\frac{1}{2}(h,h)_\nabla\right]$, and **Dirichlet inner product** in domain D

$$
\begin{aligned}
(f_1, f_2)_\nabla &:= (2\pi)^{-1} \int_D \nabla f_1(z) \cdot \nabla f_2(z) d^2 z \\
&= \mathrm{Cov}\big((h, f_1)_\nabla, (h, f_2)_\nabla\big)
\end{aligned}
$$

♢ STARRING THE GFF! *(Courtesy of N.-G. Kang)* ♢



Liouville QG

Random Measure

$$d\mu = \text{“}e^{\gamma h} d^2 z\text{”}$$

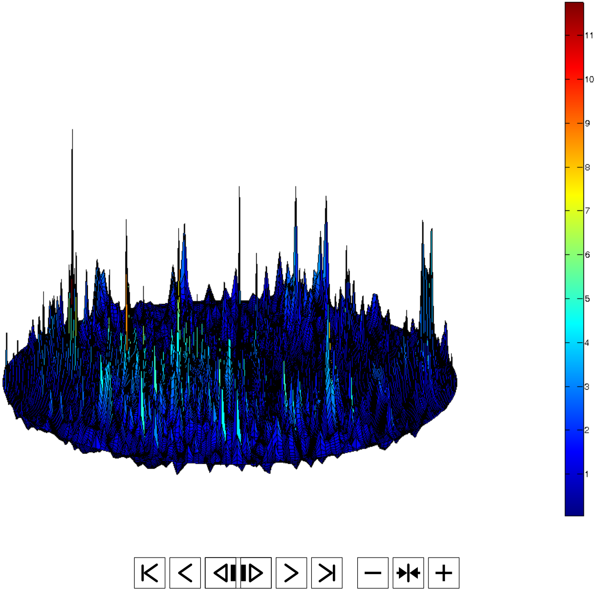
♢

The Emergence of Quantum Gravity

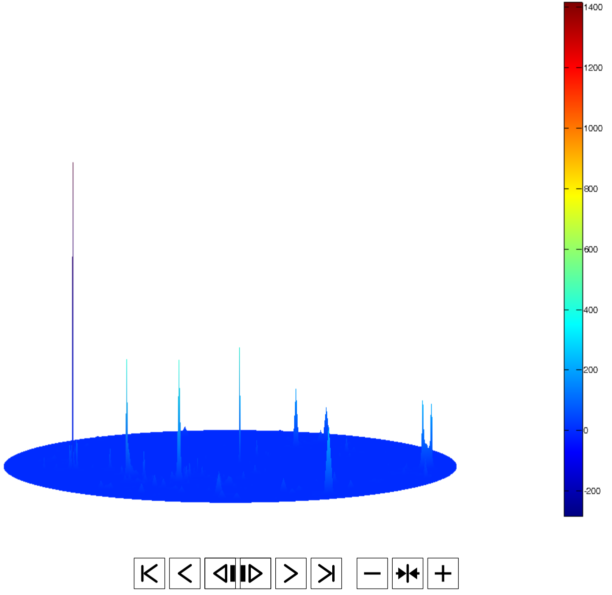
(Courtesy of N.-G. Kang)

♢

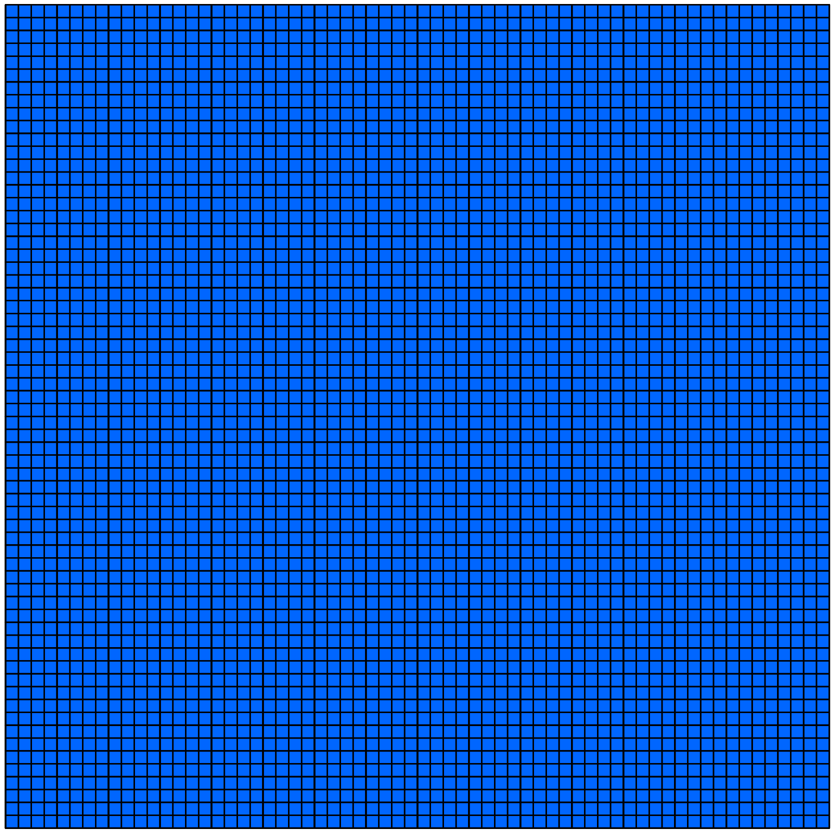
gamma = 0.5



gamma = 2

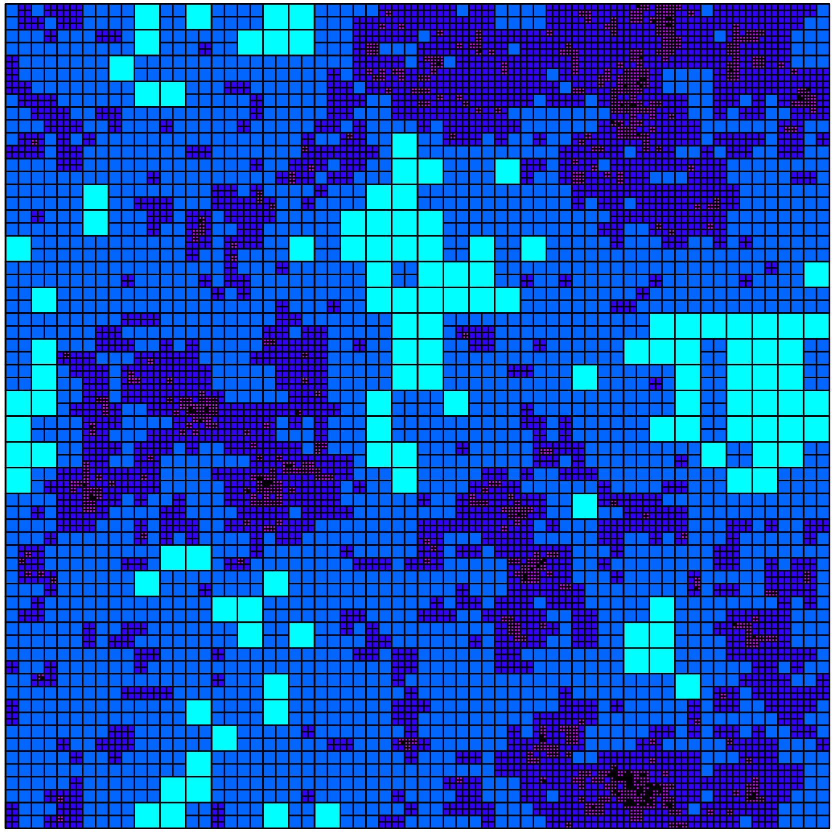


Euclidean (Lebesgue) Measure ($\gamma = 0$)



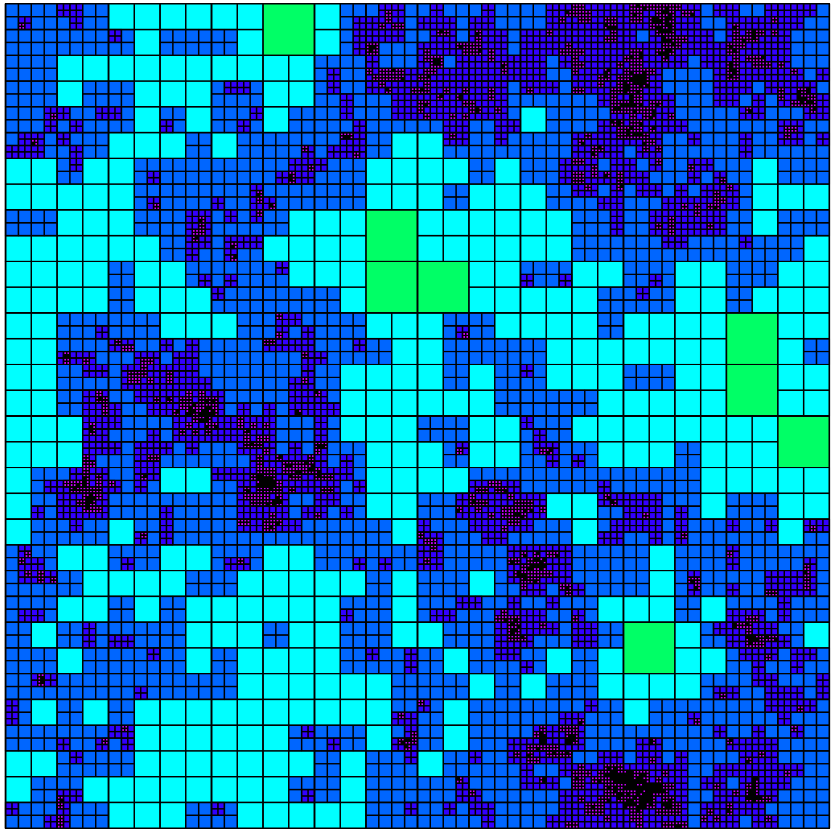
Euclidean squares of same Lebesgue area ε^2

Discrete Quantum Gravity Measure ($\gamma = 1$)



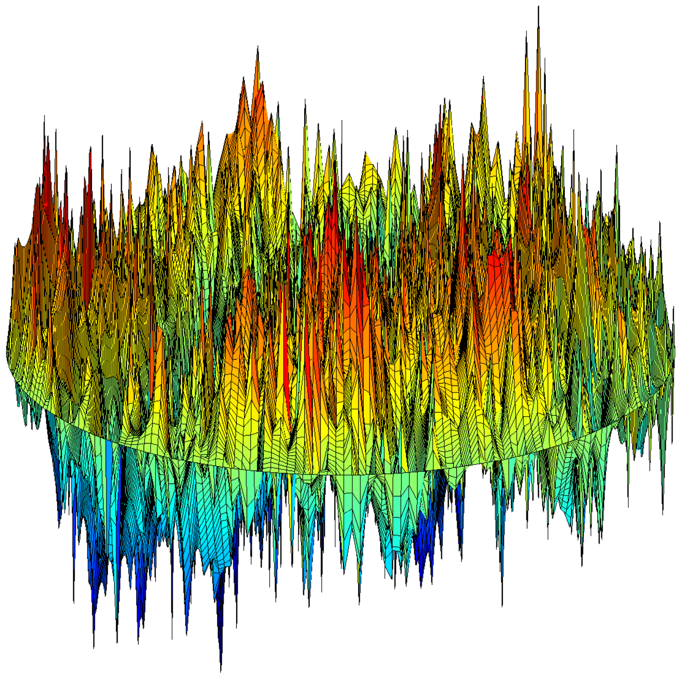
Random measure $d\mu = e^{\gamma h} d^2 z$, $\gamma = 1$ *with h discrete GFF on a fine torus lattice. Euclidean squares of similar quantum area* $\leqslant \delta$ ($= 2^{-12} \times$ *total area*).

Discrete Quantum Gravity Measure ($\gamma = 3/2$)



Euclidean squares of similar quantum area δ

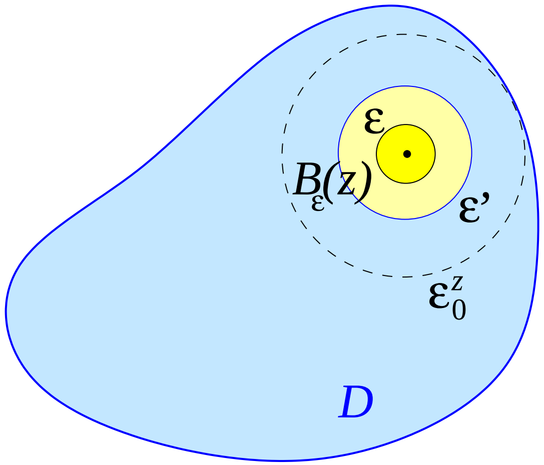
GFF Regularization & Potential Theory



(Courtesy of N.-G. Kang)

Regularization: Circular Average of the GFF

$h_\varepsilon(z)$ **mean value of** h **on circle** $\partial B_\varepsilon(z)$

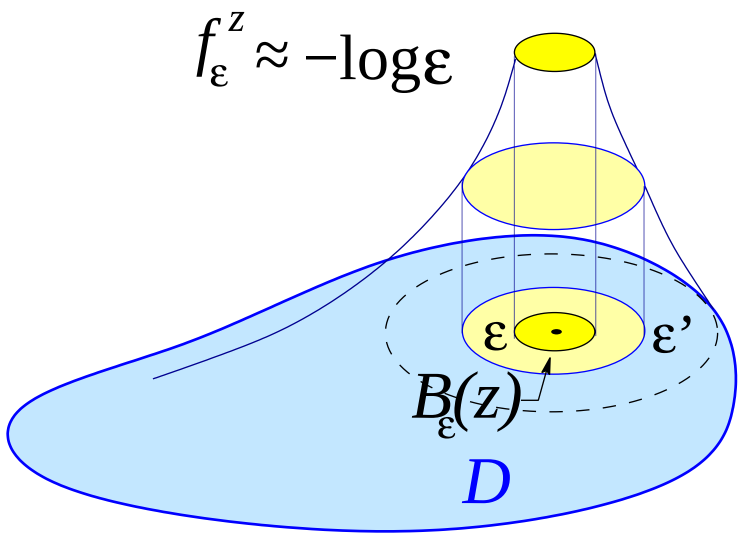


$$(h,\rho) := \int_D h(y)\rho(y)d^2y$$

$$h_\varepsilon(z) := (h,\rho_\varepsilon^z) = (h, f_\varepsilon^z)_\nabla$$

$\rho_\varepsilon^z(\cdot)$ *uniform Dirac dist. of mass* 1 *on circle* $\partial B_\varepsilon(z)$

GFF Circular Average & Logarithmic Potential

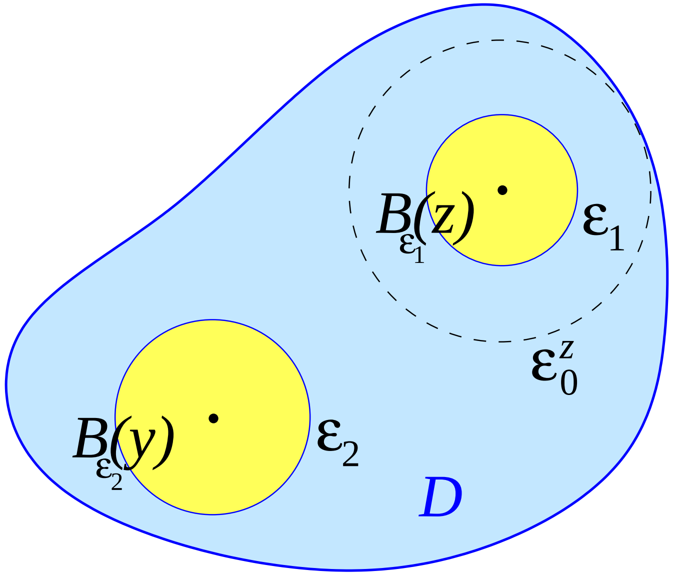


$$h_\varepsilon(z) \quad := \quad (h, f_\varepsilon^z)_\nabla$$

$$f_\varepsilon^z(\cdot) \quad := \quad -\log(|\cdot - z| \vee \varepsilon) + G_z(\cdot)$$

$G_z(\cdot)$ *harmonic extension of* $\log|\cdot - z|$ *in* D

Potential & Covariance



$$\begin{aligned} \mathrm{Cov}\big(h_{\varepsilon_1}(z), h_{\varepsilon_2}(y)\big) &= \mathrm{Cov}\left[(h, f^z_{\varepsilon_1})_\nabla, (h, f^y_{\varepsilon_2})_\nabla\right] \\ &= (f^z_{\varepsilon_1}, f^y_{\varepsilon_2})_\nabla \end{aligned}$$

$$h_\varepsilon(z) := (h, f^z_\varepsilon)_\nabla$$

$$f^z_\varepsilon(\cdot) := -\log(|\cdot - z| \vee \varepsilon) + G_z(\cdot)$$

- Regularization

$h_\varepsilon(z)$ **mean value of** h **on circle** $\partial B_\varepsilon(z)$

- Variance

$$\mathrm{Var}\, h_\varepsilon(z) = (f_\varepsilon^z, f_\varepsilon^z)_\nabla = f_\varepsilon^z(z) = \log[C(z,D)/\varepsilon]$$

$C(z,D)$ *conformal radius of* D *viewed from* z

$h_\varepsilon(z)$ **Gaussian random variable**

$$\mathbb{E}\, e^{\gamma h_\varepsilon(z)} = e^{\gamma^2 \mathrm{Var}\, h_\varepsilon(z)/2} = \left(\frac{C(z,D)}{\varepsilon}\right)^{\gamma^2/2} \qquad \square$$

Stochastic Quantum Area

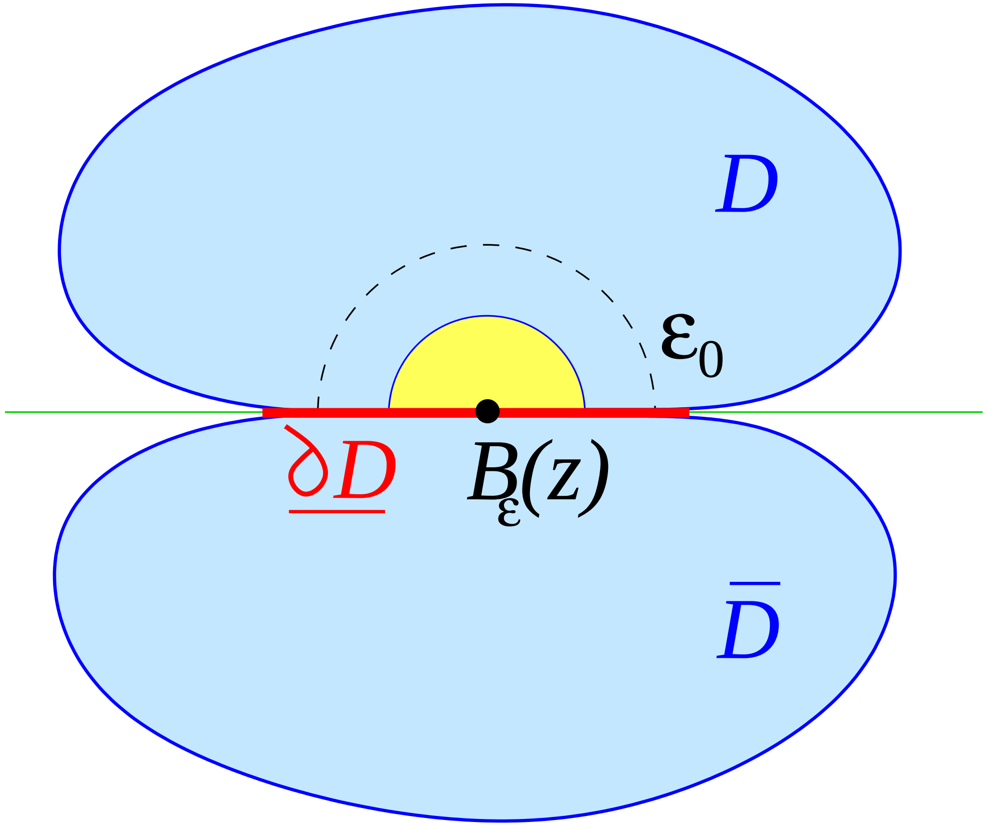
$$d\mu_\varepsilon := \exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2} d^2 z$$

converges to a random measure as $\varepsilon \to 0$ *for*

$$\gamma < 2$$

(Høegh-Krohn, '71)

Boundary Liouville Quantum Gravity



- *GFF with free boundary conditions on* $\underline{\partial D}$;
- *Half-circle averages* $\hat{h}_\varepsilon(z)$.

Quantum Area Measure

$$d\mu_\varepsilon := \exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2} d^2 z$$

converges to a random measure as $\varepsilon \to 0$ *for* $\gamma < 2$.

Quantum Boundary Measure

$$d\hat{\mu}_\varepsilon := \exp\left[\frac{\gamma}{2}\hat{h}_\varepsilon(z)\right] \varepsilon^{\gamma^2/4} dz$$

converges to a boundary random measure as $\varepsilon \to 0$ *for* $\gamma < 2$.

Liouville Quantum Gravity, KPZ & Schramm-Loewner Evolution II

Bertrand Duplantier

Institut de Physique Théorique, Saclay, France

& Scott Sheffield, MIT Math, USA

– UK-Japan Winter School 2012 –

String Theory, Geometry

& Mathematical Physics

Department of Physics

University of Oxford / 5 – 8 January 2012

KPZ RELATION

Knizhnik, Polyakov, Zamolodchikov '88

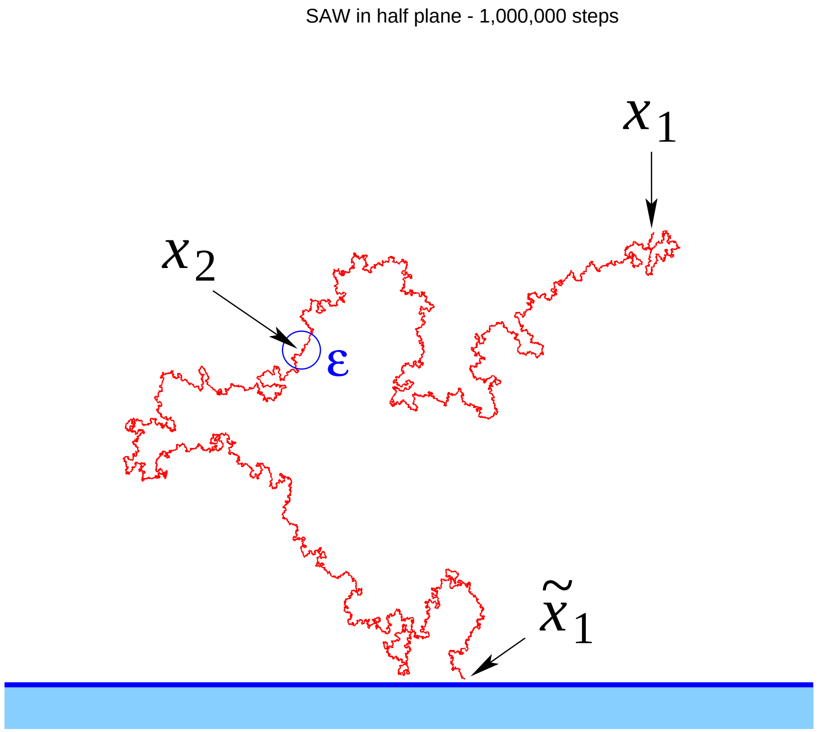
"Dynamics of 2d gravity is very rich and even now not completely explored. One of the problems was the field -dependent cut-off which one must use in order to preserve general covariance on the world sheet. I tried to overcome this difficulty by using a different gauge. I found, quite unexpectedly, the emergence of the SL(2,R) current algebra and, in a subsequent joint paper by Sasha Zamolodchikov, Dima Knizhnik and myself, this symmetry allowed us to find the fractal dimensions of minimal models dressed by the gravitational field. This work had a tragic element. Dima, my fantastically talented graduate student, died of a sudden heart failure before the work was done. I didn't even know that he was working on this subject. But after his death Sasha and I read his notes and received a crucial insight, which allowed us to finish the work."

A.M. POLYAKOV, *From Quarks to Strings*, arXiv:0812.0183.

“A few years before this work Kazakov and David suggested that the discrete version of 2d gravity can be described by the various matrix models. It was hard to be certain that these models really have a continuous limit described by the Liouville theory, there were no proofs of this conjecture. To our surprise we found that the anomalous dimensions coming from our theory coincide with the ones computed from the matrix model. That left no doubts that in the case of the minimal models the Liouville description is equivalent to the matrix one. This relation received a lot of attention.”

A.M. POLYAKOV, *From Quarks to Strings*, arXiv:0812.0183.

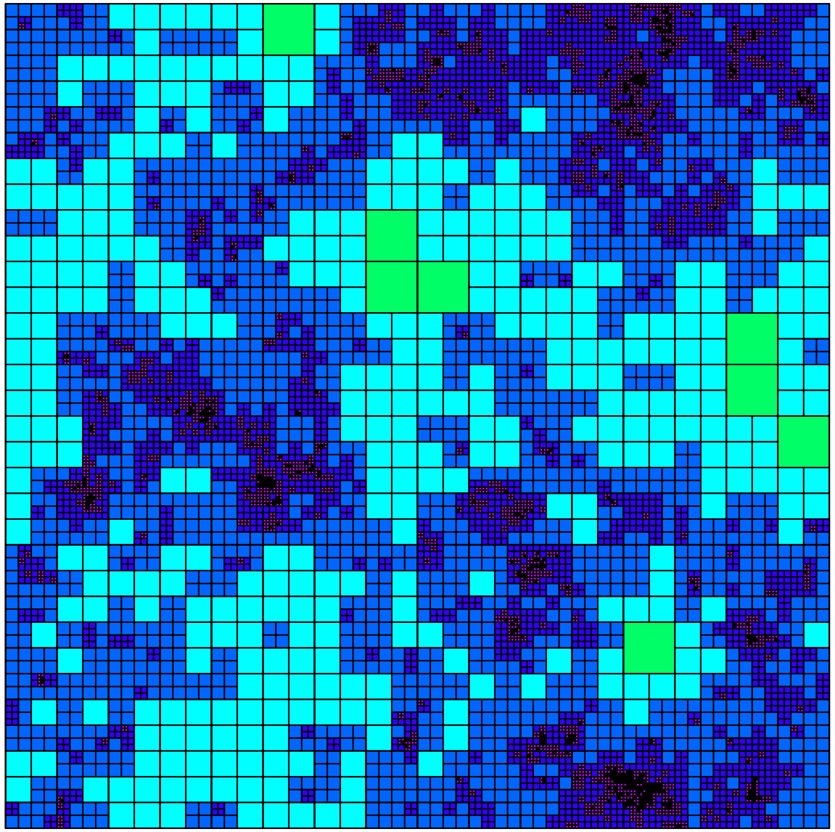
Scaling Exponents of (Random) Fractals in $\mathbb{H}$



Probabilities & Hausdorff Dimensions (e.g., SLE_κ)

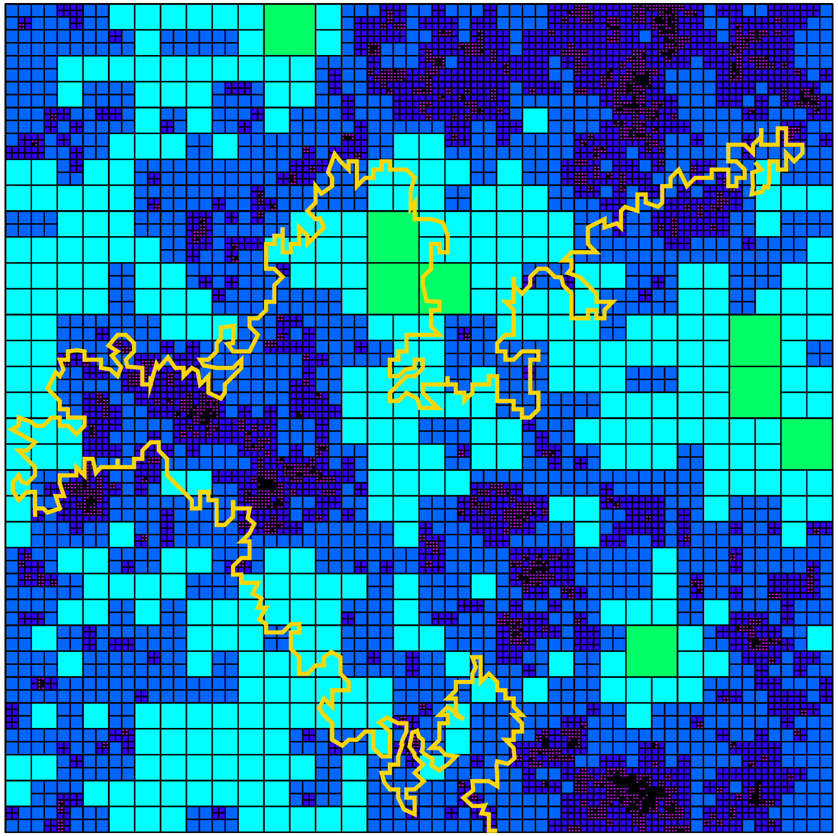
$$\mathbb{P} \asymp \varepsilon^{2x}, \quad \tilde{\mathbb{P}} \asymp \varepsilon^{\tilde{x}}, \qquad D = 2 - 2x_2 \qquad (= 1 + \kappa/8)$$

Discrete Quantum Gravity Measure ($\gamma = 3/2$)



Euclidean squares of similar quantum area δ

Quantum Gravity Scaling Exponents



$$\mathbb{P} \asymp \delta^{\Delta}, \qquad \tilde{\mathbb{P}} \asymp \tilde{\delta}^{\tilde{\Delta}}$$

KPZ '88

x and Δ ($\tilde{x}$ and $\tilde{\Delta}$) are related by the **KPZ formula**

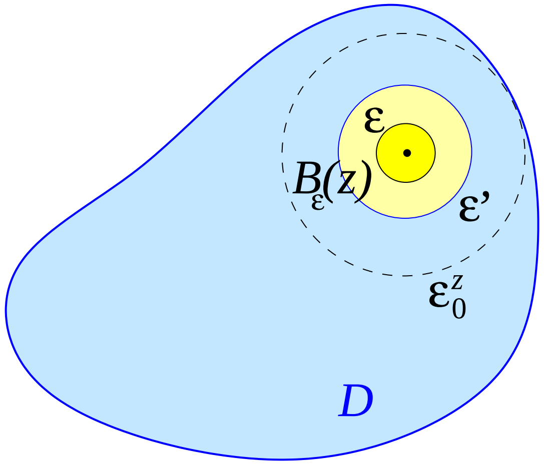
$$x = \left(1 - \frac{\gamma^2}{4}\right)\Delta + \frac{\gamma^2}{4}\Delta^2$$

KPZ is a Theorem *[D. & Sheffield, '08]*

PRL **102**, *150603 (2009) & Invent. Math.* **185**, *333 (2011)*

Kazakov '86; D. & Kostov '88 [Random matrices]
David; Distler & Kawai '88 [Liouville field theory]
Benjamini & Schramm '08; Rhodes & Vargas '11 [Math]
David & Bauer '09

GFF & Brownian Motion



- $h_\varepsilon(z)$ mean value of h on circle $\partial B_\varepsilon(z)$
- Define $t := -\log \varepsilon$, $\mathcal{B}_t := h_{\varepsilon = e^{-t}}(z)$; for z fixed, the law of $\mathcal{B}_t$ is **standard Brownian motion** in t

$$\mathrm{Var}\left[(h_\varepsilon - h_{\varepsilon'})(z)\right] = |\log(\varepsilon/\varepsilon')| = |t - t'| = \mathrm{Var}\left[\mathcal{B}_t - \mathcal{B}_{t'}\right] \ \square$$

GFF Liouville Weighted Measure

$$h_\varepsilon(z) = (h, f_\varepsilon^z)_\nabla \qquad \operatorname{Var} h_\varepsilon(z) = (f_\varepsilon^z, f_\varepsilon^z)_\nabla$$

$$\begin{aligned}
\exp\left[-\frac{1}{2}(h,h)_\nabla + \gamma(h, f_\varepsilon^z)_\nabla\right] &= \exp\left[-\frac{1}{2}(h',h')_\nabla + \frac{\gamma^2}{2}(f_\varepsilon^z, f_\varepsilon^z)_\nabla\right] \\
&= \exp\left[-\frac{1}{2}(h',h')_\nabla\right] \mathbb{E}\, e^{\gamma h_\varepsilon(z)} \\
h &\overset{\text{(in law)}}{=} h' + \gamma f_\varepsilon^z \quad (h' \text{ standard GFF}) \qquad \square \\
h_\varepsilon(z) &= \mathcal{B}_t + \gamma f_\varepsilon^z(z) \quad \bullet
\end{aligned}$$

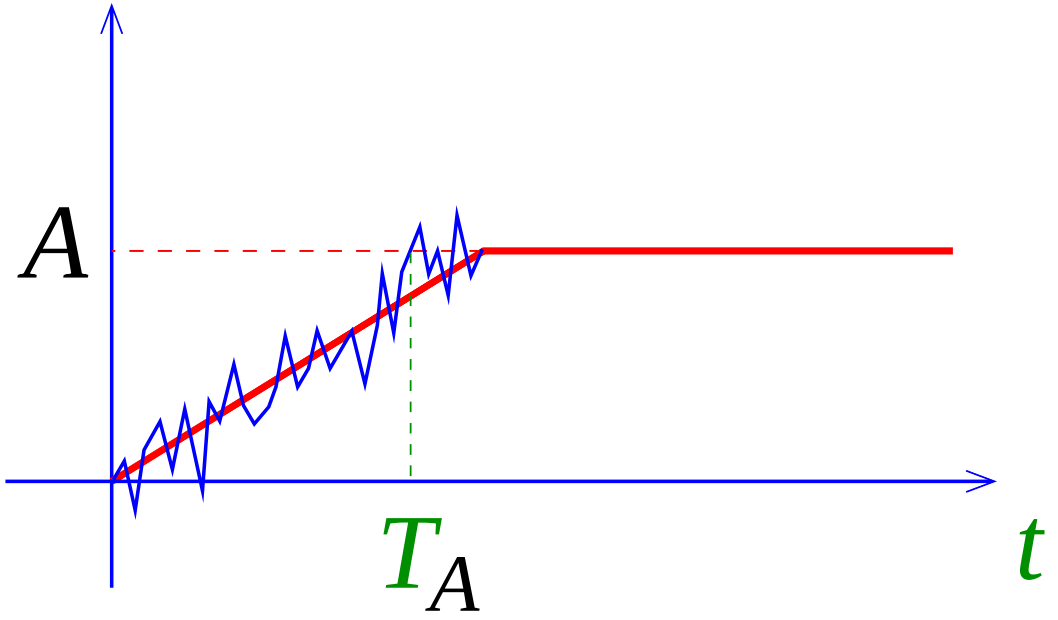
Quantum Ball & Brownian Motion

Quantum area

- $$\delta := \exp\left[\gamma h_\varepsilon(z)\right] \pi \varepsilon^{2+\gamma^2/2}$$

Given z, $h_\varepsilon(z)$ is standard Brownian motion $\mathcal{B}_t$, $t = -\log \varepsilon$,
plus the deterministic term: $-\gamma \log \varepsilon = \gamma t$

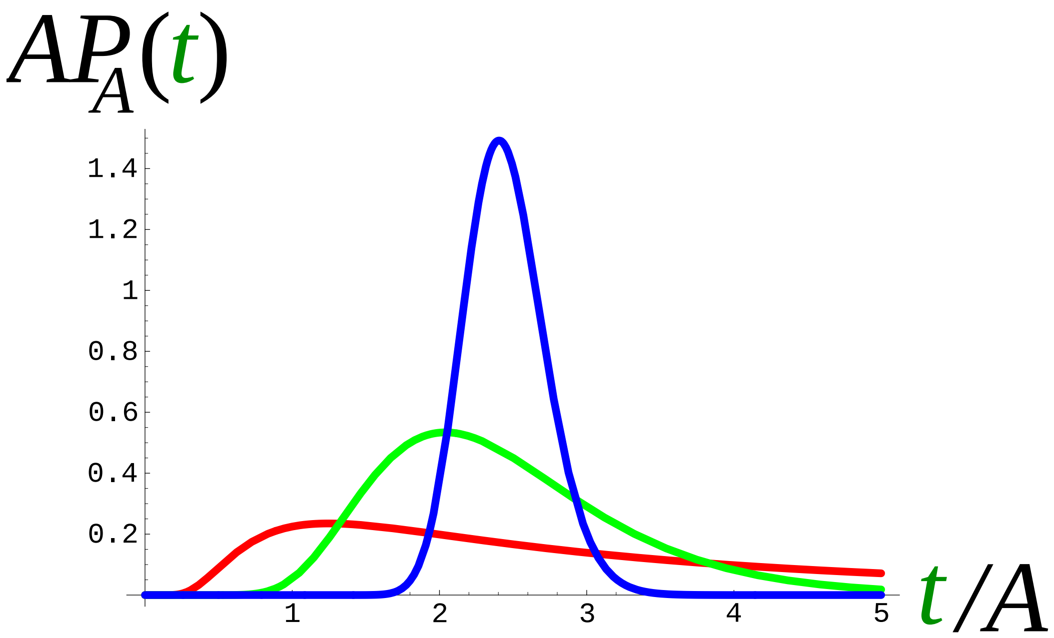
$$\begin{aligned} \delta &= \exp\left(\gamma \mathcal{B}_t - at\right), \qquad a := 2 - \gamma^2/2 \\ -\log \delta &= at - \gamma \mathcal{B}_t \qquad \square \qquad \text{(B. M. \& drift)} \end{aligned}$$



The stochastic area of ball $B_\varepsilon(z)$ equals δ at **stopping time** T_A

$$
\begin{aligned}
-\log \varepsilon_A = T_A \quad &:= \quad \inf\{t : at - \gamma \mathcal{B}_t = A\} \\
A := -\log \delta > 0,\ a \quad &= \quad 2 - \gamma^2/2 > 0 \quad (\gamma < 2)
\end{aligned}
$$

Probability Distribution $\left(\gamma = \sqrt{8/3}\right)$ $[A = 2;\ 20;\ 200]$



$$
\begin{aligned}
P_A(t)dt \quad &:= \quad \mathbb{P}\left(T_A \in [t, t+dt]\right) \\
P_A(t) \quad &= \quad \frac{A}{\sqrt{2\pi t^3}} \exp\left[-\frac{1}{2t}(A - at)^2\right]
\end{aligned}
$$

Euclidean scaling exponent

X a (random) fractal of *Euclidean scaling exponent* x (Hausdorff dimension $2-2x$):

$$\mathbb{P}\{B_\varepsilon(z)\cap X\neq\emptyset\}\asymp\varepsilon^{2x}$$

uniformly in z.

Quantum scaling exponent

Quantum scaling exponent Δ of X when (h,z) and X are sampled *independently* from the quantum gravity measure and from the law of X:

$$\mathbb{E}\,\mathbb{P}\{B_{\varepsilon_A}(z)\cap X\neq\emptyset\}\asymp\mathbb{E}\left[{\varepsilon_A}^{2x}\right]\asymp\delta^{\Delta}$$

KPZ Theorem

Stochastic probability & stopping time

$$
\begin{aligned}
-\log \varepsilon_A &= T_A = \inf\{t : at - \gamma \mathcal{B}_t = A = -\log \delta\} \\
{\varepsilon_A}^{2x} &= \exp(-2xT_A)
\end{aligned}
$$

Brownian Martingale & Large Deviations

$$
\begin{aligned}
\mathbb{E}\left[{\varepsilon_A}^{2x}\right] &= \mathbb{E}\left[e^{-2xT_A}\right] = \exp(-\Delta A) = \delta^{\Delta} \\
2x &= a\Delta + \frac{\gamma^2}{2}\Delta^2, \qquad a = 2 - \frac{\gamma^2}{2} \qquad \text{(KPZ)} \quad \square
\end{aligned}
$$

Brownian Exponential Martingale Lemma

$T_A = -\log \varepsilon_A$ is the first time t such that

$$at - \gamma \mathcal{B}_t = A,$$

$\mathcal{B}_t$ standard Brownian motion $(\mathcal{B}_0 = 0)$. Consider for any β the *Brownian exponential martingale*

$$\mathbb{E}\left[\exp(-\beta \mathcal{B}_t - \beta^2 t/2)\right] = \mathbb{E}\left[\exp(-\beta \mathcal{B}_0\right] = 1.$$

At the stopping time $t = T_A < +\infty$ *in particular:*

$$\mathbb{E}\left[\exp(-\beta\mathcal{B}_{T_A} - \beta^2 T_A/2)\right] = 1$$

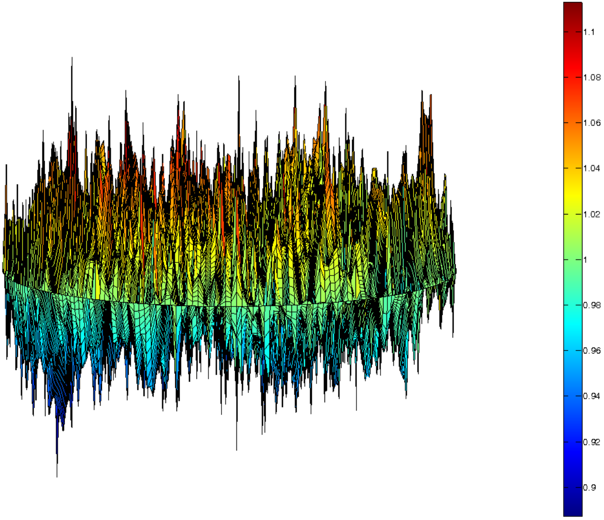
By definition $\gamma\mathcal{B}_{T_A} = aT_A - A$, whence

$$\begin{aligned} \mathbb{E}\left(\exp[-(a\beta/\gamma + \beta^2/2)T_A]\right) \quad &= \quad \exp(-\beta A/\gamma) \\ = \mathbb{E}\left[\exp(-2xT_A)\right] \qquad & \qquad 2x := a\beta/\gamma + \beta^2/2 \\ \Delta := \beta/\gamma; \qquad & \qquad = a\Delta + \frac{\gamma^2}{2}\Delta^2 \quad \square \end{aligned}$$

Liouville Quantum Duality

$$\gamma > 2, \gamma' = 4/\gamma < 2$$

gamma = 0.02



LIOUVILLE QUANTUM DUALITY

Baby-Universes: *Das, Dhar, Sengupta, Wadia '90; Jain & Mathur 92; Korchemsky '92; Alvarez-Gaumé, Barbón, Crnković '93; Durhuus '94; Ambjørn, Durhuus, Jonsson '94*

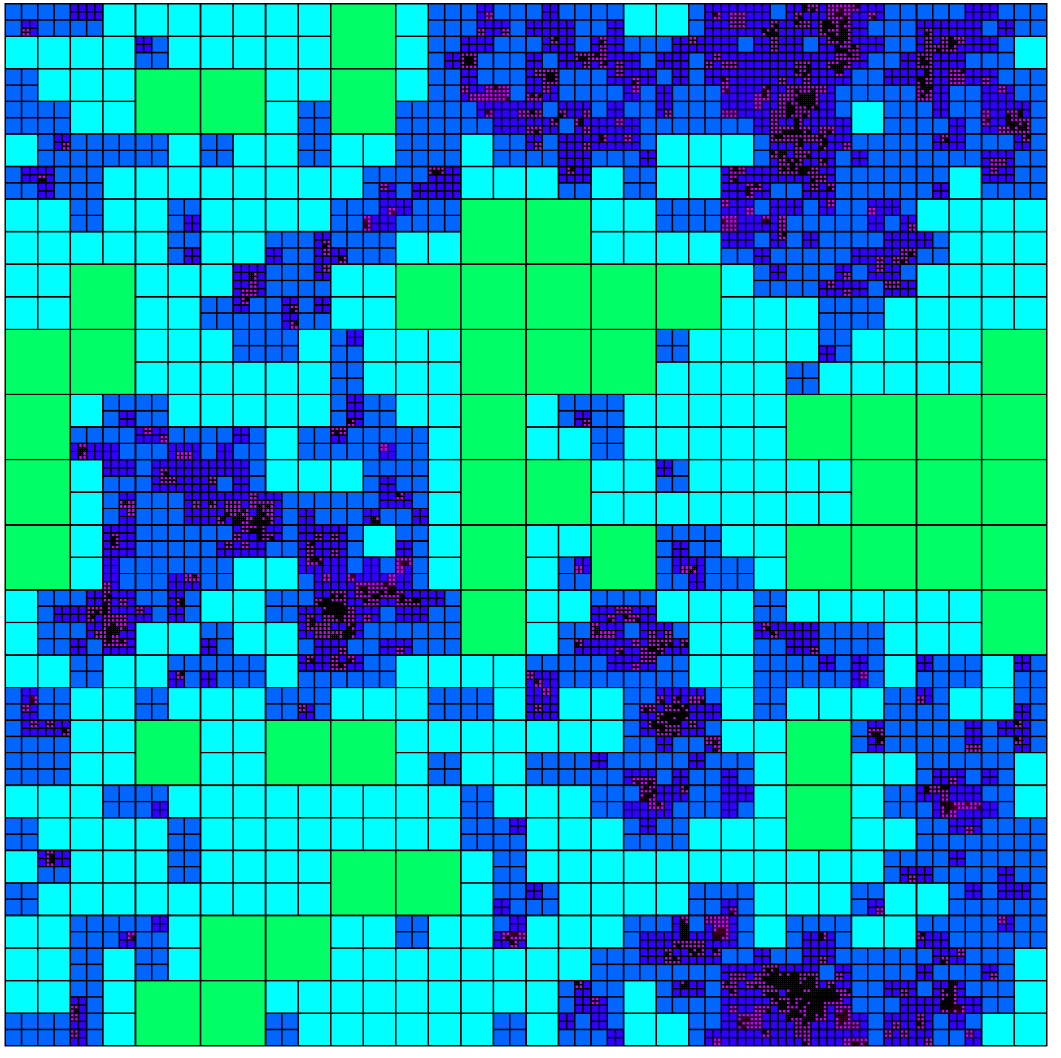
The Other Branch of Gravity, *Klebanov '95*

Dual Dimensions

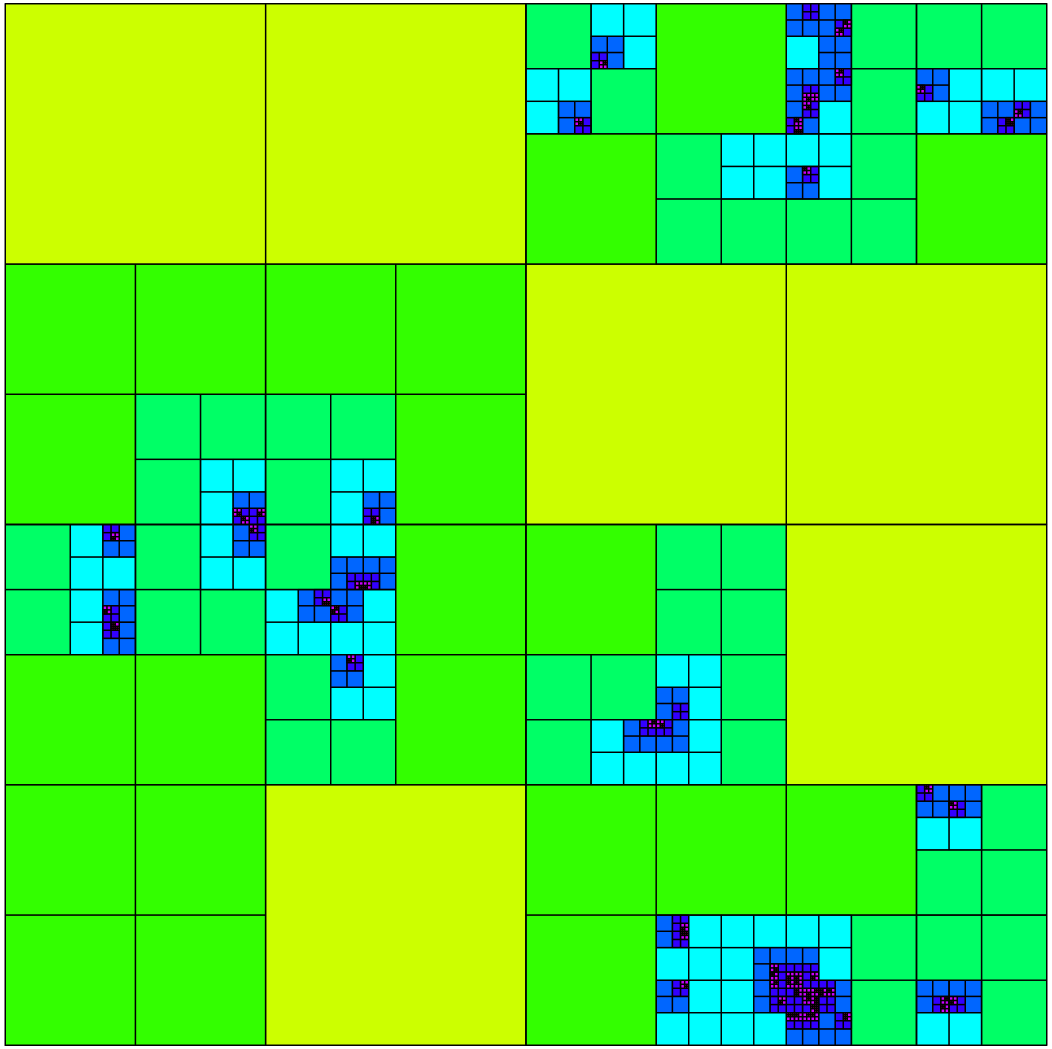
$$\begin{aligned} \gamma > 2, \gamma' &= 4/\gamma < 2 \\ \Delta_\gamma - 1 &:= \frac{4}{\gamma^2}(\Delta_{\gamma'} - 1) \end{aligned}$$

D. & Sheffield, PRL **102**, *150603 (2009)*

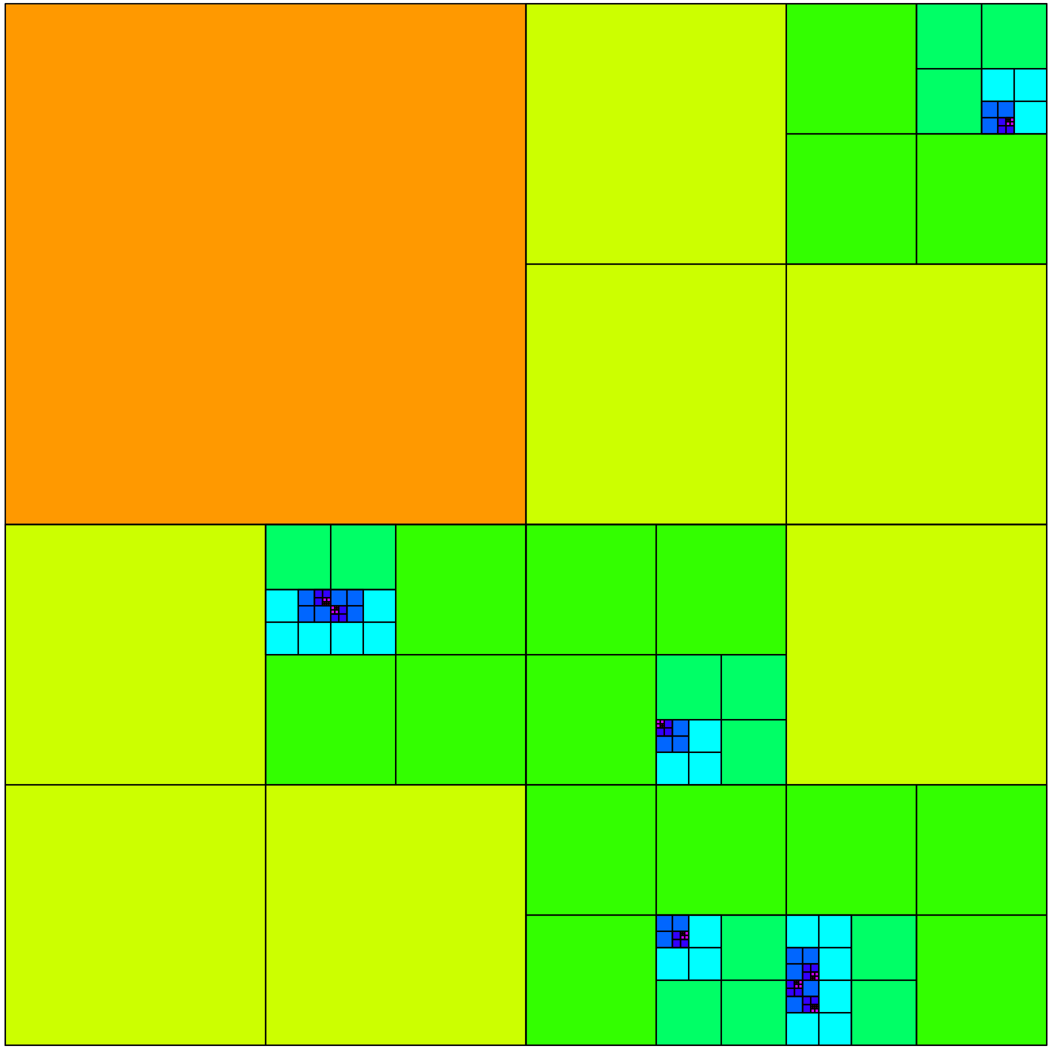
QG Measure ($\gamma = 2$)



QG Measure ($\gamma = 5$)



QG Measure ($\gamma = 10$)



Duality: $\gamma > 2, \gamma' := 4/\gamma < 2$

γ & γ'-Quantum Balls

$$Q_{\gamma'} = Q_\gamma := \frac{2}{\gamma} + \frac{\gamma}{2}$$

$$\begin{aligned} \mu_{\gamma'}(B_\varepsilon(z)) = \varepsilon^{\gamma' Q} e^{\gamma' h_\varepsilon(z)} &= \mu_\gamma(B_\varepsilon(z))^{\gamma'/\gamma} = \mu_\gamma^{4/\gamma^2} \\ \delta' &= \delta^{4/\gamma^2} \end{aligned}$$

Dual Dimensions

Ball covering of fractal X

$$
\begin{aligned}
N_{\gamma'}(\delta', X) &= N_{\gamma}(\delta, X) \\
\delta'^{\Delta_{\gamma'}-1} = \delta^{(4/\gamma^2)(\Delta_{\gamma'}-1)} &= \delta^{\Delta_{\gamma}-1} \\
\Delta_{\gamma} - 1 &:= \frac{4}{\gamma^2}(\Delta_{\gamma'} - 1)
\end{aligned}
$$

"The other branch of gravity," *I. Klebanov, '95*

Brownian Approach to Duality

$$\mathbb{E}[\exp(-2xT_A)1_{T_A<\infty}] = \exp(-\beta_\gamma A) = \delta^{\Delta_\gamma}$$

$$\beta_\gamma(x) := (a_\gamma^2 + 4x)^{1/2} - a_\gamma, \qquad \Delta_\gamma := \beta_\gamma/\gamma$$

$$a_\gamma := \frac{2}{\gamma} - \frac{\gamma}{2} < 0$$

$$\mathbb{P}(T_A < \infty) = \mathbb{E}[1_{T_A<\infty}] = \delta^{\Delta_\gamma(0)} = \delta^{1-4/\gamma^2} = \delta/\delta',$$

$$\frac{\mathbb{E}[\exp(-2xT_A)1_{T_A<\infty}]}{\mathbb{E}[1_{T_A<\infty}]} = \delta^{\Delta_\gamma} \times \frac{\delta'}{\delta} = \delta'^{\Delta_{\gamma'}}.$$