HYPERKÄHLER AND QUATERNIONIC KÄHLER GEOMETRY

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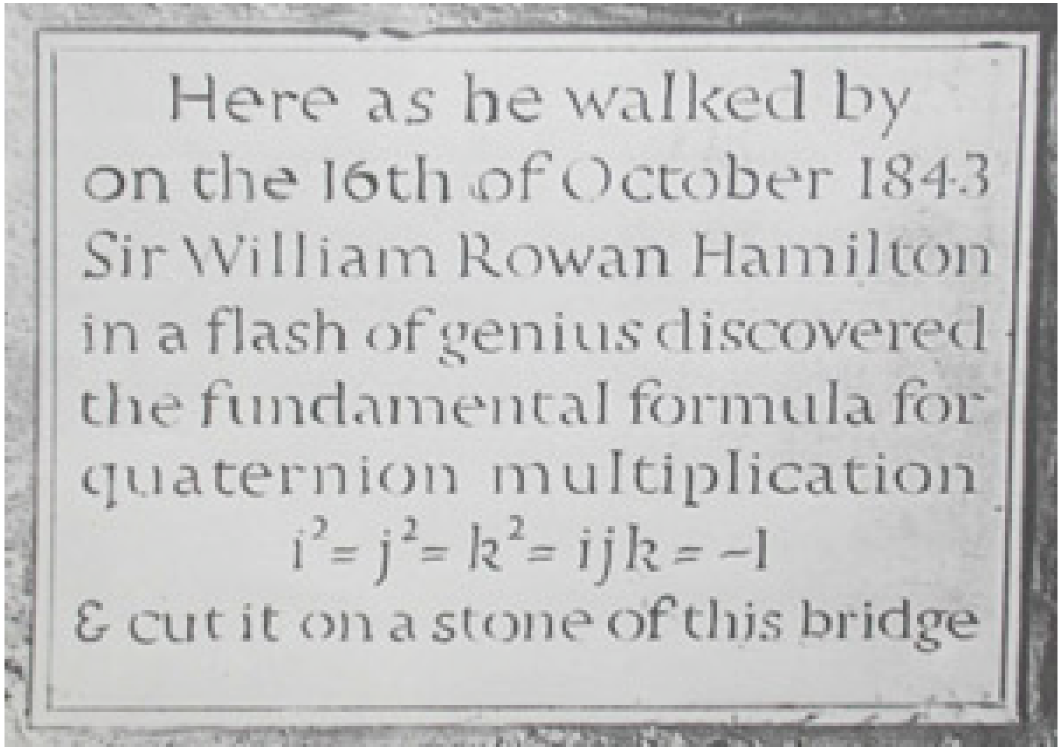
TWISTOR SPACES


Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

- $i^2 = j^2 = k^2 = ijk = -1$

- i, j, k complex structures

- $(x_1 i + x_2 j + x_3 k)^2 = -(x_1^2 + x_2^2 + x_3^2)$

- S^2 of complex structures

- hyperkähler manifold M: I, J, K

- $Z = M \times S^2$

- $T_{(m,\mathbf{x})}Z = T_m M \oplus T_{\mathbf{x}} S^2$

- S^2 complex manifold, $\mathbf{C}P^1$

- complex structure at $(m, \mathbf{x}) \in M \times S^2$ defined by $(I_{\mathbf{x}}, I)$

- $Z = M \times S^2$ complex manifold
- $Z \xrightarrow{p} \mathbf{C}P^1$ holomorphic
- $p^{-1}(\mathbf{x}) = (M, I_{\mathbf{x}})$

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- $Z \xrightarrow{p} \mathbf{C}P^1$ holomorphic
- $p^{-1}(\mathbf{x}) = (M, I_{\mathbf{x}})$
- For each $m \in M$, (m, S^2) is a holomorphic section of p
  twistor line
- $(m, \mathbf{x}) \mapsto (m, -\mathbf{x})$ antiholomorphic involution

- hyperkähler manifold M: $\omega_1, \omega_2, \omega_3$

- complex structure I:

  ω_1 Kähler form $\qquad \omega_2 + i\omega_3$ holomorphic symplectic form

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- $\omega_\zeta = (\omega_2 + i\omega_3) + 2\omega_1\zeta - (\omega_2 - i\omega_3)\zeta^2$

  (ζ coordinate on $\mathbf{C}P^1 = \mathbf{C} \cup \infty$)

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- ω_ζ holomorphic section of $\Lambda^2 T_F^*(2)$

  ($\mathcal{O}(2)$ = pull back of line bundle of degree 2 on $\mathbf{C}P^1$)

HYPERKÄHLER TWISTOR SPACE

- holomorphic fibration $Z \to \mathbf{C}P^1$
- symplectic form along the fibres
- "symplectic manifold over the field of functions of ζ"

EXAMPLE: $\mathbf{H}^n$

- twistor space $\mathbf{C}^{2n}(1) \to \mathbf{P}^1$
- $\mathbf{C}^{2n}$ symplectic vector space
- $\omega(v, w) \in \mathcal{O}(2)$

EXAMPLE: MAGNETIC MONOPOLES

- $SU(2)$ Bogomolny equations on $\mathbf{R}^3$

- $F = *\nabla\phi$

$$\|\phi\| \sim 1 - \frac{k}{2r} - \frac{Q(x,x)}{4r^5} + \ldots$$

- $4k$-dimensional hyperkähler moduli space, $SO(3)$ action rotating $\omega_1, \omega_2, \omega_3$

EXAMPLE: MONOPOLES

- complex structure $\sim$ axis in $\mathbf{R}^3 \Rightarrow$ all complex structures are equivalent under $SO(3)$

- $\mathcal{M}_k \cong$ space of rational functions R_k:

$$\frac{p(z)}{q(z)} = \frac{a_0 + a_1 z + \ldots + a_{k-1} z^{k-1}}{b_0 + b_1 z + \ldots + z^k}$$

- twistor space $Z = U_0 \cup U_1$, $U_0 \cong U_1 \cong R_k \times \mathbf{C}$

- on $U_0 \cap U_1 = R_k \times \mathbf{C}^*$

$$\tilde{\zeta} = \frac{1}{\zeta}, \quad \tilde{q}\left(\frac{z}{\zeta^2}\right) = \frac{1}{\zeta^{2k}} q(z), \quad \tilde{p}\left(\frac{z}{\zeta^2}\right) = e^{-2z/\zeta} p(z) \bmod q(z)$$

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- symplectic structure: $f_x(p/q) = p(x), g_x(p/q) = q(x)$

$$\{f_x, g_y\} = \frac{p(x)q(y) - q(x)p(y)}{x - y}$$

TWISTOR LINES

- sections of $Z \to \mathbf{C}P^1$
- denominator: $q(z) = z^k + b_{k-1}(\zeta) z^{k-1} + \ldots + b_0(\zeta)$

$$\tilde{q}\left(\frac{z}{\zeta^2}\right) = \frac{1}{\zeta^{2k}} q(z)$$

- $b_j(\zeta)$ polynomial of degree $2(k-j)$

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- $b_j(\zeta)$ polynomial of degree $2(k-j)$

- $z^k + b_{k-1}(\zeta)z^{k-1} + \ldots + b_0(\zeta) = 0$ spectral curve, genus $(k-1)^2$

- numerator:

$$\tilde{p}\left(\frac{z}{\zeta^2}\right) = e^{-2z/\zeta} p(z) \bmod q(z)$$

- on spectral curve S, $q = 0$

- $p(z, \zeta)$ non-vanishing function on $U_0 \cap S$, $\tilde{p}(z, \tilde{\zeta})$ on $U_1 \cap S$

- $\Rightarrow$ non-vanishing section of line bundle with transition function $e^{-2z/\zeta}$.

Recall

- $\mathcal{L}_X \omega_1 = 0$ moment map μ

- $i_X \omega_1 = d\mu$

- $K d\mu(Y) = -\omega_1(X, KY) = -g(IX, KY) =$

$$= g(KIX, Y) = g(JX, Y) = i_X \omega_2(Y)$$

- $dKd\mu = d(i_X \omega_2) = \mathcal{L}_X \omega_2 = \omega_3$

Kähler potential

- S^1-action rotation about direction u

- moment map

Theta function of spectral curve

$$\mu = \frac{4}{(N+1)(N+2)}\frac{\vartheta^{(N+2)}(0)}{\vartheta^{(N)}(0)} - \frac{1}{3}Q(u,u)$$

NJH, *Integrable systems in Riemannian geometry*, in Surveys in Differential Geometry Vol. 4, C.-L. Terng and K. Uhlenbeck, (eds.), International Press, Cambridge, Mass. (1999), 21– 80.

HYPERKÄHLER QUOTIENT

- complex structures I, J, K: Kähler forms $\omega_1, \omega_2, \omega_3$

- Hamiltonian group action G, moment maps μ_1, μ_2, μ_3

- $\mu = (\mu_1, \mu_2, \mu_3) : M \to \mathfrak{g}^* \otimes \mathbf{R}^3$

- $\mu^{-1}(0)/G$ is hyperkähler

TWISTOR VERSION

- G induces a (local) holomorphic G^c action on Z
- holomorphic moment map ν section of $\mathfrak{g}^*(2)$
- $\nu^{-1}(0)/G^c =$ twistor space of quotient

EXAMPLE: EGUCHI-HANSON METRIC ON T^*S^2

- flat twistor space $V(1) \oplus V^*(1) \to \mathbf{P}^1$ $\dim V = 2$

- $\mathbf{C}^*$ action $(v, \alpha) \mapsto (\lambda v, \lambda^{-1}\alpha)$

- moment section $\nu(v, \alpha) = \alpha(v) - \zeta$

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- $\nu^{-1}(0)/\mathbf{C}^* = T^*\mathbf{P}(V)$ if $\zeta = 0$

  complex structure I

$\nu^{-1}(0)/\mathbf{C}^* = \mathbf{P}(V) \times \mathbf{P}(V^*) \backslash \{\alpha(v) = 0\}$ if $\zeta \neq 0$

complex structure J (quadric surface in $\mathbf{C}^3$)

QUATERNIONIC KÄHLER TWISTOR SPACE

- rank 3 bundle of quaternions

- P principal $SO(3)$ frame bundle

- $P/SO(2)$ unit sphere bundle = bundle of complex structures

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- $T(P/SO(2)) = H \oplus V$

- complex structure $(I_{\mathbf{x}}, I)$

Recall...

- $P = SO(3)$ frame bundle
- θ_i well-defined 1-forms on P
- dim $P \times \mathbf{R}^+ = 4n + 4$
- define $\varphi_i = d(t\theta_i)$ ($t = \mathbf{R}^+$ coordinate)
- three closed 2-forms $\varphi_1, \varphi_2, \varphi_3$
- $P \times \mathbf{R}^+ =$ **Swann bundle** or **hyperkähler cone**

- $SO(3)$ action rotates I, J, K

- $SO(2)$ fixes I

- $\mathbf{C}^* = SO(2) \times \mathbf{R}^+$ acts I-holomorphically

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$$P/SO(2) = \text{Swann bundle}/\mathbf{C}^*$$

- vector field X, $X^\perp$ symplectic orthogonal wrt $\omega_2 + i\omega_3$

- $X^\perp/X$ defines a holomorphic *contact structure* on $P/SO(2)$

- contact structure on Z^{2n+1}: rank $2n$ subbundle $E \subset T$

- $0 \to E \to T \stackrel{\pi}{\to} T/E = L \to 0$

- $X, Y \in E$ then $\pi[X, Y]$ non-degenerate

- $\pi \sim \theta \in T^* \otimes L$

QUATERNIONIC KÄHLER TWISTOR SPACE

- complex manifold Z^{2n+1}
- contact form θ − holomorphic section of $T^* \otimes L$
- $\theta \wedge (d\theta)^n \neq 0$ ($\Rightarrow L^{n+1} \cong K^*$)
- $+$ family of rational curves $+$ antiholomorphic involution

EXAMPLE

- quaternionic projective space $\mathbf{H}P^n$
- Swann bundle $= \mathbf{H}^{n+1}\backslash\{0\}$
- $\mathbf{H}^{n+1}\backslash\{0\}/\mathbf{C}^* = \mathbf{C}^{2n+2}\backslash\{0\}/\mathbf{C}^* = \mathbf{C}P^{2n+1}$

QUATERNIONIC KÄHLER QUOTIENT

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- $\Rightarrow$ "moment" section ν of $\mathfrak{g}^* \otimes L$

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- twistor space of quotient is $\nu^{-1}(0)/G^c$

TWISTOR FORMALISM

- hyperkähler geometry = holomorphic symplectic geometry over $\mathbf{C}[[\zeta]]$

- quaternionic geometry = holomorphic contact geometry

CIRCLE ACTIONS ON QK MANIFOLDS

- twistor space $Z^{2n+1} = P/SO(2) \to M^{4n}$
- $U(1)$ action on M, preserving QK structure
- moment section of L

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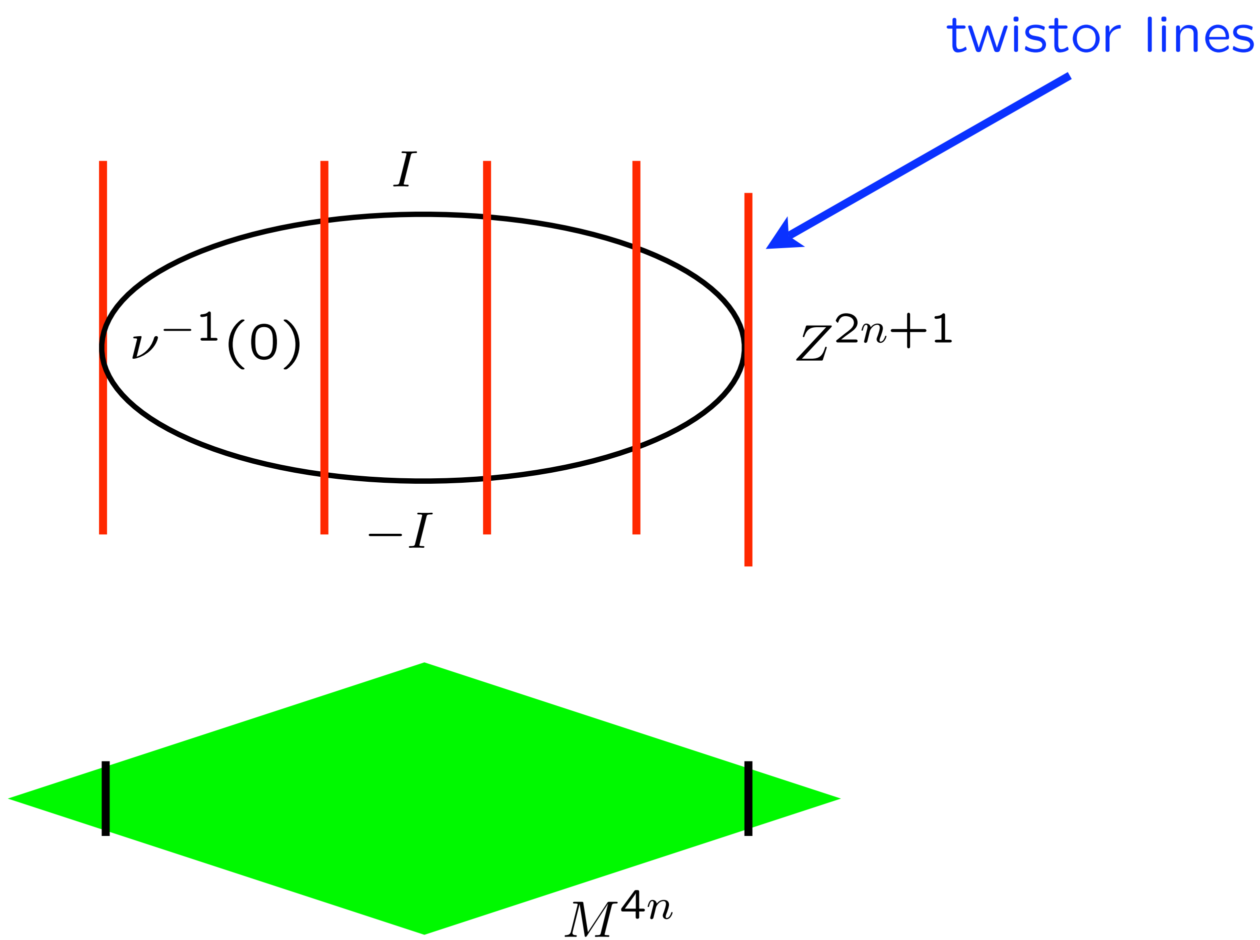
- $L^{n+1} \cong K^* \Rightarrow$ degree 2 on each fibre

- $\nu^{-1}(0)$ intersects a generic fibre in two antipodal points



twistor lines
I
$\nu^{-1}(0)$
Z^{2n+1}
$-I$
M^{4n}

- outside the branch locus, defines a complex structure on M

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- $\nu^{-1}(0) = D^{+} + D^{-}$

- $D^{+} - D^{-}$ divisor class, degree 0 on each fibre

- $\Rightarrow$ holomorphic line bundle on Z

- $\Rightarrow$ $U(1)$ connection on the canonical bundle of (M, I)

Recall

- $SL(n, \mathbf{H}) \cdot U(1)$ action of $\mathbf{C}$ on tangent bundle T

- if a torsion-free connection ∇ preserves this structure, it is unique

- **complex quaternionic** – complex manifold

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- $L \cong \mathcal{O}(2)$ $\qquad \nu^{-1}(0) =$ quadric surface $= \mathbf{C}P^1 \times \mathbf{C}P^1$

- real lines $\mathbf{R}P^1 \subset \mathbf{C}P^1$

- $\mathbf{H}P^1 \backslash \mathbf{R}P^1 \cong H^2 \times S^2$ (scalar-flat Kähler)

HYPERHOLOMORPHIC BUNDLES

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- A is hyperholomorphic if F is of type $(1,1)$ with respect to I, J, K ($\Rightarrow$ all complex structures)

- $\dim M = 4$ anti-self-dual

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- connection A on a principal G-bundle over M, curvature $F \in \Omega^2(\mathfrak{g})$

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- **Prop**: A hyperholomorphic connection corresponds to a holomorphic bundle on the twistor space which is trivial on each real twistor line.

EXAMPLE: QUOTIENTS

- $\mu : M \to \mathfrak{g}^* \otimes \mathbf{R}^3$

- If G acts properly and freely on $\mu^{-1}(0)$ then...

- ... the quotient metric on $\mu^{-1}(0)/G$ is **hyperkähler**...

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- $\mu : M \to \mathfrak{g}^* \otimes \mathbf{R}^3$
- If G acts properly and freely on $\mu^{-1}(0)$ then...
- ... the quotient metric on $\mu^{-1}(0)/G$ is **hyperkähler**...
- $\mu^{-1}(0)$ is a principal G-bundle over the quotient

- $P = \mu^{-1}(0) \subset M$ has an induced metric
- orthogonals to G-orbits $\Rightarrow$ connection on P...
- .... which is hyperholomorphic

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- $\nu^{-1}(0)$ principal G^c bundle over twistor space Z

GAUGE-THEORETIC QUOTIENTS

- $\mathcal{G}$ = gauge transformations = sections of $P \times_G G \to M$

- choose $x \in M$

- evaluation homomorphism $g \in \mathcal{G} \mapsto g(x) \in G$

- hyperholomorphic connection on a G-bundle over moduli space $\sim$ universal bundle on $M \times \mathcal{M}$

EXAMPLE: MONOPOLE DIRAC OPERATOR

- Dirac operator $\mathbf{D} = i\nabla_1 + j\nabla_2 + k\nabla_3 - \phi$
- E vector bundle associated to some representation of $SU(2)$
- $\mathbf{D} : \mathbf{C}^\infty(S \otimes E) \to \mathbf{C}^\infty(S \otimes E)$

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- $E =$ adjoint rep, connection is Levi-Civita on the tangent bundle

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- $\mathbf{D}^* : \Omega^{1,0}(E) \oplus \Omega^{0,1}(E) \to \Omega^{1,1}(E) \oplus \Omega^{1,1}(E)$

- $\mathbf{D}^*(\psi_1, \psi_2) = (\nabla^{0,1}\psi_1 - \Phi\psi_2, \nabla^{1,0}\psi_2 - \Phi^*\psi_1)$

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- complex structure I: hypercohomology of $\mathcal{O}(E) \stackrel{\Phi}{\to} \mathcal{O}(E \otimes K)$
- complex structure J: de Rham cohomology of flat connection

NEXT LECTURE

- M hyperkähler
- circle action fixing ω_1, rotating ω_2, ω_3
- $\Rightarrow$ natural hyperholomorphic line bundle