CITY UNIVERSITY
LONDON

Introduction to non-Hermitian Hamiltonian systems with PT symmetry, applications to integrable systems

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Outline

Why is Hermiticity a good property to have?

- Hermiticity ensures real energies
  Schrödinger equation $H\psi = E\psi$

$$\left.\begin{aligned} \langle\psi| H |\psi\rangle &= E \langle\psi|\psi\rangle \\ \langle\psi| H^\dagger |\psi\rangle &= E^* \langle\psi|\psi\rangle \end{aligned}\right\} \Rightarrow 0 = (E - E^*) \langle\psi|\psi\rangle$$

- Hermiticity ensures conservation of probability densities

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$$

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)| e^{iH^\dagger t} e^{-iHt} |\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle$$

- Thus when $H \neq H^\dagger$ one usually thinks of dissipation.
- However, these systems are usually open and do not possess a self-consistent description.

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- Thus when $H \neq H^\dagger$ one usually thinks of dissipation.
- However, these systems are usually open and do not possess a self-consistent description.

Hermiticity is not essential

- Operators $\mathcal{O}$ which are left invariant under an antilinear involution $\mathcal{I}$ and whose eigenfunctions Φ also respect this symmetry,

$$[\mathcal{O}, \mathcal{I}] = 0 \quad \wedge \quad \mathcal{I}\Phi = \Phi$$

have a real eigenvalue spectrum.
[E. Wigner, *J. Math. Phys.* 1 (1960) 409]

- By defining a new metric also a consistent quantum mechanical framework has been developed for theories involving such operators.
[F. Scholtz, H. Geyer, F. Hahne, *Ann. Phys.* 213 (1992) 74, C. Bender, S. Boettcher, *Phys. Rev. Lett.* 80 (1998) 5243, A. Mostafazadeh, *J. Math. Phys.* 43 (2002) 2814]

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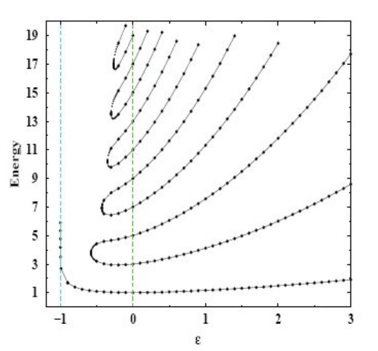
There are plenty of well studied examples of non-Hermitian systems in the literature

"Recent" classical example

$$\mathcal{H} = \frac{1}{2}p^2 + x^2(ix)^\varepsilon \qquad \text{for } \varepsilon \geq 0$$



[C.M. Bender, S. Boettcher, *Phys. Rev. Lett.* 80 (1998) 5243]

A more classical example

- Lattice Reggeon field theory:

$$\mathcal{H} = \sum_{\vec{\imath}} \left[\Delta a^{\dagger}_{\vec{\imath}} a_{\vec{\imath}} + iga^{\dagger}_{\vec{\imath}}(a_{\vec{\imath}} + a^{\dagger}_{\vec{\imath}})a_{\vec{\imath}} + \tilde{g} \sum_{\vec{\jmath}} (a^{\dagger}_{\vec{\imath}+\vec{\jmath}} - a^{\dagger}_{\vec{\imath}})(a_{\vec{\imath}+\vec{\jmath}} - a_{\vec{\imath}}) \right]$$

- $a^{\dagger}_{\vec{\imath}}, a_{\vec{\imath}}$ are creation and annihilation operators, $\Delta, g, \tilde{g} \in \mathbb{R}$
[J.L. Cardy, R. Sugar, *Phys. Rev.* D12 (1975) 2514]

- for one site this is almost ix^3

$$\begin{aligned} \mathcal{H} &= \Delta a^{\dagger} a + iga^{\dagger}\left(a + a^{\dagger}\right) a \\ &= \frac{1}{2}\left(\hat{p}^2 + \hat{x}^2 - 1\right) + i\frac{g}{\sqrt{2}}(\hat{x}^3 + \hat{p}^2\hat{x} - 2\hat{x} + i\hat{p}) \end{aligned}$$

with $a = (\omega\hat{x} + i\hat{p})/\sqrt{2\omega}$, $a^{\dagger} = (\omega\hat{x} - i\hat{p})/\sqrt{2\omega}$
[P. Assis and A.F., J. Phys. A41 (2008) 244001]

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- quantum spin chains: (c=-22/5 CFT)

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N} \sigma_i^x + \lambda\sigma_i^z\sigma_{i+1}^z + ih\sigma_i^z \qquad \lambda, h \in \mathbb{R}$$

[G. von Gehlen, J. Phys. A24 (1991) 5371]

- affineToda field theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{\beta^2}\sum_{k=a}^{\ell} n_k \exp(\beta\alpha_k\cdot\phi)$$

$a = 1 \equiv$ conformal field theory (Lie algebras)
$a = 0 \equiv$ massive field theory (Kac-Moody algebras)
$\beta \in \mathbb{R} \equiv$ no backscattering
$\beta \in i\mathbb{R} \equiv$ backscattering (Yang-Baxter, quantum groups)

- strings on $AdS_5 \times S^5$-background
  [A. Das, A. Melikyan, V. Rivelles, *JHEP* 09 (2007) 104]

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- deformed space-time structure
  - deformed Heisenberg canonical commutation relations

$$aa^\dagger - q^2 a^\dagger a = q^{g(N)}, \qquad \text{with } N = a^\dagger a$$

$$X = \alpha a^\dagger + \beta a, \quad P = i\gamma a^\dagger - i\delta a, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

$$[X, P] = i\hbar q^{g(N)}(\alpha\delta + \beta\gamma) + \frac{i\hbar(q^2 - 1)}{\alpha\delta + \beta\gamma}\left(\delta\gamma X^2 + \alpha\beta P^2 + i\alpha\delta XP - i\beta\gamma PX\right)$$

- limit: $\beta \to \alpha$, $\delta \to \gamma$, $g(N) \to 0$, $q \to e^{2\tau\gamma^2}$, $\gamma \to 0$

$$[X, P] = i\hbar\left(1 + \tau P^2\right)$$

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- with the standard inner product X is not Hermitian

$$X^\dagger = X + 2\tau i\hbar P \quad \text{and} \quad P^\dagger = P$$

- $\Rightarrow H(X,P)$ is in general not Hermitian
- example harmonic oscillator:

$$\begin{aligned} H_{ho} &= \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2, \\ &= \frac{p_0^2}{2m} + \frac{m\omega^2}{2}(1+\tau p_0^2)x_0(1+\tau p_0^2)x_0, \\ &= \frac{p_0^2}{2m} + \frac{m\omega^2}{2}\left[(1+\tau p_0^2)^2 x_0^2 + 2i\hbar\tau p_0(1+\tau p_0^2)x_0\right]. \end{aligned}$$

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- "dynamical" noncommutative space-time
  Replace

$$[x_0, y_0] = i\theta, \qquad [x_0, p_{x_0}] = i\hbar, \qquad [y_0, p_{y_0}] = i\hbar,$$
$$[p_{x_0}, p_{y_0}] = 0, \qquad [x_0, p_{y_0}] = 0, \qquad [y_0, p_{x_0}] = 0,$$

with $\theta \in \mathbb{R}$, by

$$\begin{aligned} [X, Y] &= i\theta(1 + \tau Y^2) & [X, P_x] &= i\hbar(1 + \tau Y^2) \\ [Y, P_y] &= i\hbar(1 + \tau Y^2) & [X, P_y] &= 2i\tau Y(\theta P_y + \hbar X) \\ [P_x, P_y] &= 0 & [Y, P_x] &= 0 \end{aligned}$$

$\Rightarrow$ Non-Hermitian representation

$$X = (1 + \tau y_0^2)x_0 \quad Y = y_0 \quad P_x = p_{x_0} \quad P_y = (1 + \tau y_0^2)p_{y_0}$$

$$X^\dagger = X + 2i\tau\theta Y \quad Y^\dagger = Y \quad P_y^\dagger = P_y - 2i\tau\hbar Y \quad P_x^\dagger = P_x$$

[A.F., L. Gouba, F. Scholtz, J.Phys. A43 (2010) 345401]
[A.F., L. Gouba, B. Bagchi, J.Phys. A43 (2010) 425202]

How to explain the reality of the spectrum?

1. Pseudo/Quasi-Hermiticity
2. Supersymmetry (Darboux transformations)
3. $\mathcal{PT}$-symmetry

Pseudo/Quasi-Hermiticity

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \quad \Leftrightarrow \quad H^\dagger \rho = \rho H \qquad \rho = \eta^\dagger \eta \quad (*)$$

	$H^\dagger = \rho H \rho^{-1}$	$H^\dagger \rho = \rho H$	$H^\dagger = \rho H \rho^{-1}$
positivity of ρ	$\checkmark$	$\checkmark$	$\times$
ρ Hermitian	$\checkmark$	$\checkmark$	$\checkmark$
ρ invertible	$\checkmark$	$\times$	$\checkmark$
terminology	(*)	quasi-Herm.	pseudo-Herm.
spectrum of H	real	could be real	real
definite metric	guaranteed	guaranteed	not conclusive

- quasi-Hermiticity: [J. Dieudonné, Proc. Int. Symp. (1961) 115] [F. Scholtz, H. Geyer, F. Hahne, Ann. Phys. 213 (1992) 74]
- pseudo-Hermiticity: [M. Froissart, Nuovo Cim. 14 (1959) 197] [A. Mostafazadeh, J. Math. Phys. 43 (2002) 2814]

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positivity of ρ	$\checkmark$	$\checkmark$	$\times$
ρ Hermitian	$\checkmark$	$\checkmark$	$\checkmark$
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	$H^{\dagger} = \rho H \rho^{-1}$	$H^{\dagger}\rho = \rho H$	$H^{\dagger} = \rho H \rho^{-1}$
positivity of ρ	✓	✓	×
ρ Hermitian	✓	✓	✓
ρ invertible	✓	×	✓
terminology	(*)	quasi-Herm.	pseudo-Herm.
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Pseudo/Quasi-Hermiticity

$$h = \eta H \eta^{-1} = h^{\dagger} = (\eta^{-1})^{\dagger} H^{\dagger} \eta^{\dagger} \quad \Leftrightarrow \quad H^{\dagger}\rho = \rho H \qquad \rho = \eta^{\dagger}\eta \quad (*)$$

	$H^{\dagger} = \rho H \rho^{-1}$	$H^{\dagger}\rho = \rho H$	$H^{\dagger} = \rho H \rho^{-1}$
positivity of ρ	✓	✓	×
ρ Hermitian	✓	✓	✓
ρ invertible	✓	×	✓
terminology	(*)	quasi-Herm.	pseudo-Herm.
spectrum of H	real	could be real	real
definite metric	guaranteed	guaranteed	not conclusive

- quasi-Hermiticity: [J. Dieudonné, Proc. Int. Symp. (1961) 115] [F. Scholtz, H. Geyer, F. Hahne, Ann. Phys. 213 (1992) 74]
- pseudo-Hermiticity: [M. Froissart, Nuovo Cim. 14 (1959) 197] [A. Mostafazadeh, J. Math. Phys. 43 (2002) 2814]

Supersymmetry (Darboux transformation)

Decompose Hamiltonian $\mathcal{H}$ as:

$$\mathcal{H} = H_+ \oplus H_- = Q\tilde{Q} \oplus \tilde{Q}Q$$

- intertwining operators: $QH_- = H_+Q$ and $\tilde{Q}H_+ = H_-\tilde{Q}$

$$\Rightarrow \qquad [\mathcal{H}, Q] = [\mathcal{H}, \tilde{Q}] = 0$$

- realization: $Q = \frac{d}{dx} + W$ and $\tilde{Q} = -\frac{d}{dx} + W$

$$\Rightarrow \qquad H_\pm = -\Delta + W^2 \pm W' = -\Delta + V_\pm$$

- ground state: $H_-\Phi_n^- = \varepsilon_n \Phi_n^-$ and $H_-\Phi_m^- = 0$
  $\Rightarrow$isospectral Hamiltonians

$$H_\pm^m = -\Delta + V_\pm^m + E_m \quad H_\pm^m \Phi_n^\pm = E_n \Phi_n^\pm \quad \text{for } n > m$$

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Unbroken $\mathcal{PT}$-symmetry guarantees real eigenvalues (QM)

- $\mathcal{PT}$-symmetry: $\mathcal{PT}: \quad x \to -x \quad p \to p \quad i \to -i$

  $(\mathcal{P}: x \to -x, p \to -p; \ \mathcal{T}: x \to x, p \to -p, i \to -i)$
- $\mathcal{PT}$ is an anti-linear operator:

$$\mathcal{PT}(\lambda\Phi + \mu\Psi) = \lambda^*\mathcal{PT}\Phi + \mu^*\mathcal{PT}\Psi \qquad \lambda, \mu \in \mathbb{C}$$

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$$[\mathcal{H}, \mathcal{PT}] = 0 \quad \wedge \quad \mathcal{PT}\Phi = \Phi \qquad \Rightarrow \varepsilon = \varepsilon^* \ \text{ for } \mathcal{H}\Phi = \varepsilon\Phi$$

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How to formulate a quantum mechanical framework?

1. orthogonality
2. observables
3. uniqueness
4. technicalities (new metric etc)

Orthogonality

- Take h to be a Hermitian and diagonalisable Hamiltonian:

$$\langle \phi_n | h\phi_m \rangle = \langle h\phi_n | \phi_m \rangle$$

$$\left. \begin{array}{ll} \langle \phi_n | h\phi_m \rangle & = \varepsilon_m \langle \phi_n | \phi_m \rangle \\ \langle h\phi_n | \phi_m \rangle & = \varepsilon_n^* \langle \phi_n | \phi_m \rangle \end{array} \right\} \Rightarrow 0 = (\varepsilon_m - \varepsilon_n^*) \langle \phi_n | \phi_m \rangle$$

$$\Rightarrow \qquad n = m: \ \varepsilon_n = \varepsilon_n^* \qquad n \neq m: \ \langle \phi_n | \phi_m \rangle = 0$$

- Take H to be a non-Hermitian Hamiltonian:

$$H |\Phi_n\rangle = \varepsilon_n |\Phi_n\rangle$$

- reality and orthogonality no longer guaranteed. Define

$$\langle \Phi_n | \Phi_m \rangle_\eta := \langle \Phi_n | \eta^2 \Phi_m \rangle$$

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H is Hermitian with respect to new metric

- Assume pseudo-Hermiticity:

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Proof:

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H is Hermitian with respect to new metric

- Assume pseudo-Hermiticity:

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[Bender, Brody, Jones, Phys. Rev. Lett. 89 (2002) 270401]

$$\langle\Psi|\,\Phi\rangle_{\mathcal{CPT}} := \left(\mathcal{CPT}\,|\Psi\rangle\right)^T \cdot |\Phi\rangle$$

- In position space: $\mathcal{C}(x,y) = \sum_n \Phi_n(x)\Phi_n(y)$
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Observables

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$$\langle \Phi_n | \mathcal{O} \Phi_m \rangle_\eta = \langle \mathcal{O} \Phi_n | \Phi_m \rangle_\eta$$

$$\mathcal{O} = \eta^{-1} o \eta \qquad \Leftrightarrow \qquad \mathcal{O}^\dagger = \rho \mathcal{O} \rho^{-1}$$

  - o is an observable in the Hermitian system
  - $\mathcal{O}$ is an observable in the non-Hermitian system

- Ambiguities:
  Given H the metric is not uniquely defined for unknown h.
  $\Rightarrow$ Given only H the observables are not uniquely defined.
  This is different in the Hermitian case.
  - Fixing one more observable achieves uniqueness.
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General technique:

- Given $H \begin{cases} \text{either} & \text{solve } \eta H \eta^{-1} = h \quad \text{for} \quad \eta \Rightarrow \rho = \eta^{\dagger}\eta \\ \text{or} & \text{solve } H^{\dagger} = \rho H \rho^{-1} \quad \text{for} \quad \rho \Rightarrow \eta = \sqrt{\rho} \end{cases}$
- involves complicated commutation relations
- often this can only be solved perturbatively

Note:

- Thus, this is not re-inventing or disputing the validity of quantum mechanics.
- We only give up the restrictive requirement that Hamiltonians have to be Hermitian.

[C. Bender, *Rep. Prog. Phys.* 70 (2007) 947]
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Ising quantum spin chain of length N

$$\mathcal{H} = -\frac{1}{2}\sum_{i=1}^{N}(\sigma_i^z + \lambda\sigma_i^x\sigma_{i+1}^x + i\kappa\sigma_i^x) \qquad \kappa, \lambda \in \mathbb{R}$$

in a magnetic field in the z-direction and in a longitudinal imaginary field in the x-direction

- $\mathcal{H}$ acts on the Hilbert space of the form $(\mathbb{C}^2)^{\otimes N}$
- $\sigma_i^{x,y,z} := \mathbb{I} \otimes \mathbb{I} \otimes \ldots \otimes \sigma^{x,y,z} \otimes \ldots \otimes \mathbb{I} \otimes \mathbb{I}$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $\mathcal{H}$ is a perturbation of the $\mathcal{M}_{5,2}$-model (c=-22/5) in the $\mathcal{M}_{p,q}$-series of minimal conformal field theories
- non-unitary for $p - q > 1 \Rightarrow$ non-Hermitian Hamiltonians

[G. von Gehlen, J. Phys. A24 (1991) 5371]

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$$\mathcal{H}_{XXZ} = \frac{1}{2}\sum_{i=1}^{N-1}\left[(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta_+(\sigma_i^z\sigma_{i+1}^z - 1)\right] + \frac{\Delta_-}{2}(\sigma_1^z - \sigma_N^z),$$

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These possibilities reflect the ambiguities in the observables.

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$\mathcal{PT}$-symmetry $\Rightarrow$ domains in the parameter space of λ and κ

Broken and unbroken $\mathcal{PT}$-symmetry

$$[\mathcal{PT}, \mathcal{H}] = 0 \quad \bigwedge \quad \mathcal{PT}\Phi(\lambda,\kappa) \begin{cases} = \Phi(\lambda,\kappa) & \text{for } (\lambda,\kappa) \in U_{\mathcal{PT}} \\ \neq \Phi(\lambda,\kappa) & \text{for } (\lambda,\kappa) \in U_{b\mathcal{PT}} \end{cases}$$

$(\lambda,\kappa) \in U_{\mathcal{PT}} \Rightarrow$ real eigenvalues
$(\lambda,\kappa) \in U_{b\mathcal{PT}} \Rightarrow$ eigenvalues in complex conjugate pairs

- The two site Hamiltonian

$$\begin{aligned}
\mathcal{H} &= -\frac{1}{2}\left[\sigma_1^z + \sigma_2^z + 2\lambda\sigma_1^x\sigma_2^x + i\kappa\left(\sigma_2^x + \sigma_1^x\right)\right] \\
&= -\frac{1}{2}\left[\sigma^z \otimes \mathbb{I} + \mathbb{I} \otimes \sigma^z + 2\lambda\sigma^x \otimes \sigma^x + i\kappa\left(\mathbb{I} \otimes \sigma^x + \sigma^x \otimes \mathbb{I}\right)\right] \\
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\end{aligned}$$

with periodic boundary condition $\sigma_{N+1}^x = \sigma_1^x$

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char. polynomial factorises into 1st and 3rd order
discriminant: $\Delta = r^2 - q^3$

$$q = \frac{1}{9}\left(-3\kappa^2 + 4\lambda^2 + 3\right), \quad r = \frac{\lambda}{27}\left(18\kappa^2 + 8\lambda^2 + 9\right)$$

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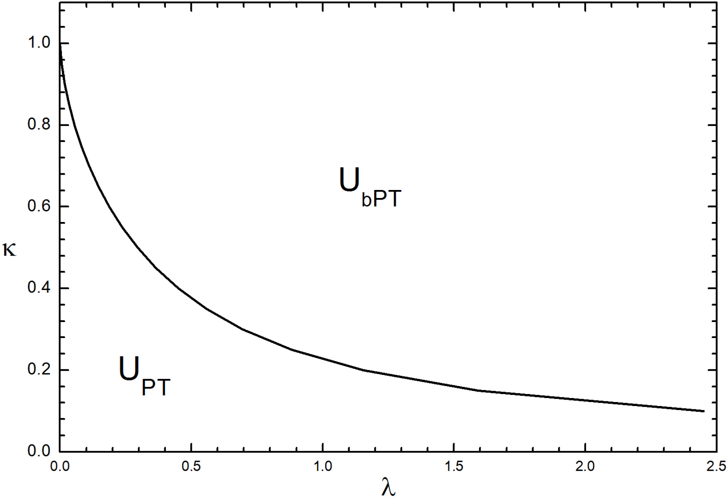
Deformed quantum spin chains (Exact Results, $N = 2$)

$$U_{\mathcal{PT}} = \left\{ \lambda, \kappa : \kappa^6 + 8\lambda^2\kappa^4 - 3\kappa^4 + 16\lambda^4\kappa^2 + 20\lambda^2\kappa^2 + 3\kappa^2 - \lambda^2 \leq 1 \right\}$$

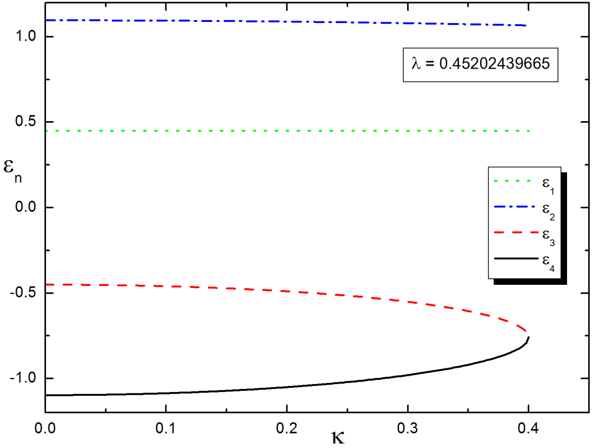


Deformed quantum spin chains (Exact Results, $N = 2$)

Real eigenvalues: $\left[\theta = \arccos\left(r/q^{3/2}\right)\right]$

$$\varepsilon_1 = \lambda, \quad \varepsilon_2 = 2q^{\frac{1}{2}}\cos\left(\frac{\theta}{3}\right) - \frac{\lambda}{3}, \quad \varepsilon_{3/4} = 2q^{\frac{1}{2}}\cos\left(\frac{\theta}{3} + \pi \mp \frac{1\pi}{3}\right) - \frac{\lambda}{3}$$

Avoided level crossing:



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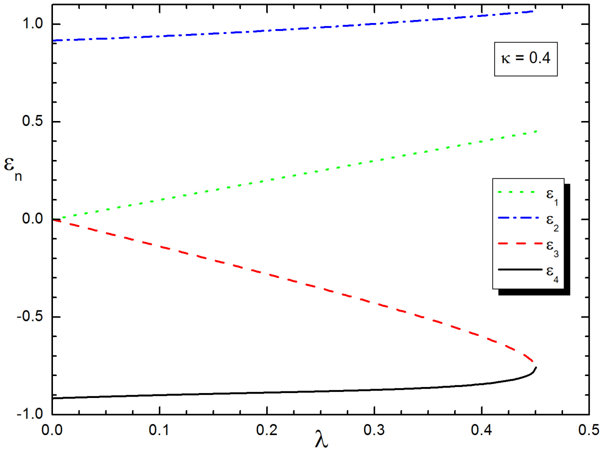

- Right eigenvectors of $\mathcal{H}$:

$$|\Phi_1\rangle = (0, -1, -1, 0) \quad |\Phi_n\rangle = (\gamma_n, -\alpha_n, -\alpha_n, \beta_n) \quad n = 2, 3, 4$$

$$\alpha_n = i\kappa\,(\lambda - \varepsilon_n + 1)$$
$$\beta_n = \kappa^2 + 2\lambda^2 + 2\lambda\varepsilon_n$$
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- signature: $s = (+, -, +, -)$

$$\mathcal{P}\,|\Phi_n\rangle = s_n\,|\Psi_n\rangle$$

from relating left and right eigenvectors

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from relating left and right eigenvectors

- $\mathcal{C}$-operator:

$$
\begin{aligned}
\mathcal{C} &= \sum_n s_n \left|\Phi_n\right\rangle \left\langle\Psi_n\right| \\
&= \begin{pmatrix} C_5 & -C_3 & -C_3 & C_4 \\ -C_3 & -C_1 - 1 & -C_1 & C_2 \\ -C_3 & -C_1 & -C_1 - 1 & C_2 \\ C_4 & C_2 & C_2 & 2(C_1 + 1) - C_5 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= \frac{\alpha_4^2}{N_4^2} - \frac{\alpha_2^2}{N_2^2} - \frac{\alpha_3^2}{N_3^2} - \frac{1}{2}, \qquad & C_2 &= \frac{\alpha_4\beta_4}{N_4^2} - \frac{\alpha_2\beta_2}{N_2^2} - \frac{\alpha_3\beta_3}{N_3^2}, \\
C_3 &= \frac{\alpha_2\gamma_2}{N_2^2} + \frac{\alpha_3\gamma_3}{N_3^2} - \frac{\alpha_4\gamma_4}{N_4^2}, \qquad & C_4 &= \frac{\beta_2\gamma_2}{N_2^2} + \frac{\beta_3\gamma_3}{N_3^2} - \frac{\beta_4\gamma_4}{N_4^2}, \\
C_5 &= \frac{\gamma_2^2}{N_2^2} + \frac{\gamma_3^2}{N_3^2} - \frac{\gamma_4^2}{N_4^2}
\end{aligned}
$$

$N_1 = \sqrt{2}$, $N_n = \sqrt{2\alpha_n^2 + \beta_n^2 + \gamma_n^2}$ for $n =, 2, 3, 4$

- metric operator:

$$\rho = \mathcal{PC} = \begin{pmatrix} C_5 & -C_3 & -C_3 & C_4 \\ C_3 & 1+C_1 & C_1 & -C_2 \\ C_3 & C_1 & 1+C_1 & -C_2 \\ C_4 & C_2 & C_2 & 2(1+C_1)-C_5 \end{pmatrix}$$

- since $i\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$
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- EV of ρ:

$$y_1 = y_2 = 1, \qquad y_{3/4} = 1 + 2C_1 \pm 2\sqrt{C_1(1+C_1)}$$

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- square root of the metric operator:

$$\eta = \rho^{1/2} = UD^{1/2}U^{-1}$$

where $D = \mathrm{diag}(y_{1,} y_{2,}, y_{3,}, y_4)$, $U = \{r_1, r_2, r_3, r_4\}$

$$\left|r_1\right\rangle = (0, -1, 1, 0)$$
$$\left|r_2\right\rangle = (C_4, 0, 0, 1 - C_5),$$
$$\left|r_{3/4}\right\rangle = (\tilde{\gamma}_{3/4}, \tilde{\alpha}_{3/4}, \tilde{\alpha}_{3/4}, \tilde{\beta}_{3/4})$$

$$\tilde{\alpha}_{3/4} = y_{3/4}(C_3 C_4 + C_2(-4C_1 + C_5 - 1))/2 - C_3 C_4$$
$$\tilde{\beta}_{3/4} = -C_3^2 - C_1 - C_1 C_5 + \left(C_3^2 + C_1(4C_1 - C_5 + 3)\right) y_{3/4},$$
$$\tilde{\gamma}_{3/4} = C_1 C_4 - C_2 C_3 + (C_2 C_3 + C_1 C_4) y_{3/4}$$

- isospectral Hermitian counterpart:

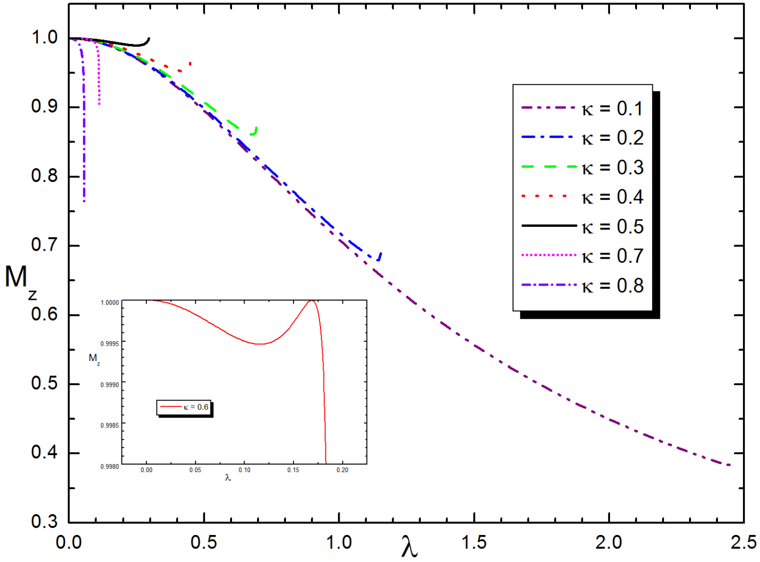
$$
\begin{aligned}
h &= \eta \mathcal{H} \eta^{-1} \\
&= \mu_1 \sigma_x \otimes \sigma_x + \mu_2 \sigma_y \otimes \sigma_y + \mu_3 \sigma_z \otimes \sigma_z + \mu_4 (\sigma_z \otimes \mathbb{I} + \mathbb{I} \otimes \sigma_z)
\end{aligned}
$$

$\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{R}$

for $\lambda = 0.1, \kappa = 0.5$:

$$
h = \begin{pmatrix}
-0.829536 & 0 & 0 & -0.0606492 \\
0 & -0.0341687 & -0.1341687 & 0 \\
0 & -0.1341687 & -0.0341687 & 0 \\
-0.0606492 & 0 & 0 & 0.897873
\end{pmatrix}
$$

The magnetization in the z-direction for $N = 2$:



Deformed quantum spin chains ($N \neq 2$, perturbation theory)

- Perturbation theory about the Hermitian part

$$H(\lambda, \kappa) = h_0(\lambda) + i\kappa h_1 \qquad h_0 = h_0^\dagger, h_1 = h_1^\dagger \quad \kappa \in \mathbb{R}$$

assume $\eta = \eta^\dagger = e^{q/2} \Rightarrow$ solve for q

$$H^\dagger = e^q H e^{-q} = H + [q, H] + \frac{1}{2}[q, [q, H]] + \frac{1}{3!}[q, [q, [q, H]]] + \cdots$$

for $c_q^{(\ell+1)}(h_0) = [q, \ldots [q, [q, h_0]] \ldots] = 0$ closed formulae:

$$h = h_0 + \sum_{n=1}^{[\frac{\ell}{2}]} \frac{(-1)^n E_n}{4^n (2n)!} c_q^{(2n)}(h_0) \quad H = h_0 - \sum_{n=1}^{[\frac{\ell+1}{2}]} \frac{\kappa_{2n-1}}{(2n-1)!} c_q^{(2n-1)}(h_0)$$

$E_n \equiv$ Euler numbers, e.g. $E_1 = 1$, $E_2 = 5$, $E_3 = 61, \ldots$

$$\kappa_n = \frac{1}{2^n} \sum_{m=1}^{[(n+1)/2]} (-1)^{n+m} \binom{n}{2m} E_m$$

$\kappa_1 = 1/2, \kappa_3 = -1/4, \kappa_5 = 1/2, \kappa_7 = -17/8, \ldots$

[C. F. de Morisson Faria, A.F., J. Phys. A39 (2006) 9269]

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$$h = h_0 + \sum_{n=1}^{[\frac{\ell}{2}]} \frac{(-1)^n E_n}{4^n (2n)!} c_q^{(2n)}(h_0) \quad H = h_0 - \sum_{n=1}^{[\frac{\ell+1}{2}]} \frac{\kappa_{2n-1}}{(2n-1)!} c_q^{(2n-1)}(h_0)$$

$E_n \equiv$ Euler numbers, e.g. $E_1 = 1$, $E_2 = 5$, $E_3 = 61, \ldots$

$$\kappa_n = \frac{1}{2^n} \sum_{m=1}^{[(n+1)/2]} (-1)^{n+m} \binom{n}{2m} E_m$$

$\kappa_1 = 1/2, \kappa_3 = -1/4, \kappa_5 = 1/2, \kappa_7 = -17/8, \ldots$

[C. F. de Morisson Faria, A.F., J. Phys. A39 (2006) 9269]

- Perturbation theory about the Hermitian part

$$H(\lambda,\kappa) = h_0(\lambda) + i\kappa h_1 \qquad h_0 = h_0^\dagger, h_1 = h_1^\dagger \quad \kappa \in \mathbb{R}$$

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further assumption

$$q = \sum_{k=1}^{\infty} \kappa^{2k-1} q_{2k-1}$$

solve recursively:

$$\begin{aligned} [h_0, q_1] &= 2ih_1 \\ [h_0, q_3] &= \frac{i}{6}[q_1, [q_1, h_1]] \\ [h_0, q_5] &= \frac{i}{6}[q_1, [q_3, h_1]] + \frac{i}{6}[q_3, [q_1, h_1]] - \frac{i}{360}[q_1, [q_1, [q_1, [q_1, h_1]]]] \end{aligned}$$

Here

$$h_0(\lambda) = -\sum\nolimits_{i=1}^{N} (\sigma_i^z + \lambda \sigma_i^x \sigma_{i+1}^x)/2, \qquad h_1 = -\sum\nolimits_{i=1}^{N} \sigma_i^x/2$$

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$$H(\lambda, \kappa) = h_0(\kappa) + \lambda h_1 \qquad h_0 \neq h_0^\dagger, h_1 = h_1^\dagger \quad \lambda \in \mathbb{R}$$

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exact result for $N = 2$:

$\lambda = 0.1$, $\kappa = 0.5$:

$$h = \begin{pmatrix} -0.829536 & 0 & 0 & -0.0606492 \\ 0 & -0.0341687 & -0.1341687 & 0 \\ 0 & -0.1341687 & -0.0341687 & 0 \\ -0.0606492 & 0 & 0 & 0.897873 \end{pmatrix}$$

$\lambda = 0.9$, $\kappa = 0.1$:

$$h = \begin{pmatrix} -0.985439 & 0 & 0 & -0.890532 \\ 0 & -0.0094167 & -0.909417 & 0 \\ 0 & -0.909417 & -0.0094167 & 0 \\ -0.890532 & 0 & 0 & 1.00427 \end{pmatrix}$$

perturbative result 4th order for $N = 2$:

$\lambda = 0.1$, $\kappa = 0.5$:

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- new notation:

$$S^N_{a_1 a_2 \ldots a_p} := \sum_{k=1}^{N} \sigma_k^{a_1} \sigma_{k+1}^{a_2} \ldots \sigma_{k+p-1}^{a_p}, \qquad a_i = x, y, z, u; i = 1, \ldots, p \leq N$$

with $\sigma^u = \mathbb{I}$ to allow for non-local interactions

- for instance:

$$\begin{aligned} H(\lambda, \kappa) &= -\frac{1}{2} \sum_{j=1}^{N} (\sigma_j^z + \lambda \sigma_j^x \sigma_{j+1}^x + i\kappa \sigma_j^x), \qquad \lambda, \kappa \in \mathbb{R} \\ &= -\frac{1}{2}(S_z^N + \lambda S_{xx}^N) - i\kappa \frac{1}{2} S_x^N \end{aligned}$$

- perturbative result for $N = 3$:

$$\begin{aligned} h &= \mu_{xx}^3(\lambda,\kappa) S_{xx}^3 + \mu_{yy}^3(\lambda,\kappa) S_{yy}^3 + \mu_{zz}^3(\lambda,\kappa) S_{zz}^3 + \mu_z^3(\lambda,\kappa) S_z^3 \\ &\quad + \mu_{xxz}^3(\lambda,\kappa) S_{xxz}^3 + \mu_{yyz}^3(\lambda,\kappa) S_{yyz}^3 + \mu_{zzz}^3(\lambda,\kappa) S_{zzz}^3 \end{aligned}$$

- perturbative result for $N = 4$:

$$\begin{aligned} h \;=\; & \mu^4_{xx}(\lambda,\kappa)S^4_{xx} + \nu^4_{xx}(\lambda,\kappa)S^4_{xux} + \mu^4_{yy}(\lambda,\kappa)S^4_{yy} + \nu^4_{yy}(\lambda,\kappa)S^4_{yuy} \\ & + \mu^4_{zz}(\lambda,\kappa)S^4_{zz} + \nu^4_{zz}(\lambda,\kappa)S^4_{zuz} + \mu^4_{z}(\lambda,\kappa)S^4_{z} + \mu^4_{xzx}(\lambda,\kappa)S^4_{xzx} \\ & + \mu^4_{xxz}(\lambda,\kappa)(S^4_{xxz} + S^4_{zxx}) + \mu^4_{yyz}(\lambda,\kappa)(S^4_{yyz} + S^4_{zyy}) \\ & + \mu^4_{yzy}(\lambda,\kappa)S^4_{yzy} + \mu^4_{zzz}(\lambda,\kappa)S^4_{zzz} + \mu^4_{xxxx}(\lambda,\kappa)S^4_{xxxx} \\ & + \mu^4_{yyyy}(\lambda,\kappa)S^4_{yyyy} + \mu^4_{zzzz}(\lambda,\kappa)S^4_{zzzz} + \mu^4_{xxyy}(\lambda,\kappa)S^4_{xxyy} \\ & + \mu^4_{xyxy}(\lambda,\kappa)S^4_{xyxy} + \mu^4_{zzyy}(\lambda,\kappa)S^4_{zzyy} + \mu^4_{zyzy}(\lambda,\kappa)S^4_{zyzy} \\ & + \mu^4_{xxzz}(\lambda,\kappa)S^4_{xxzz} + \mu^4_{xzxz}(\lambda,\kappa)S^4_{xzxz} \end{aligned}$$

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Three possibilities to obtain PT-invariant Calogero models

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Calogero-Moser-Sutherland models (extended)

$$\mathcal{H}_{BK} = \frac{p^2}{2} + \frac{\omega^2}{2}\sum_i q_i^2 + \frac{g^2}{2}\sum_{i\neq k}\frac{1}{(q_i - q_k)^2} + i\tilde{g}\sum_{i\neq k}\frac{1}{(q_i - q_k)}p_i$$

with $g, \tilde{g} \in \mathbb{R}, q, p \in \mathbb{R}^{\ell+1}$
[B. Basu-Mallick, A. Kundu, Phys. Rev. B62 (2000) 9927]

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2. Other potentials apart from the rational one?
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[A. F., Mod. Phys. Lett. A21 (2006) 691, Acta P. 47 (2007) 44]

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$$\mathcal{H}_\mu = \frac{1}{2}(p+i\mu)^2 + \frac{1}{2}\sum\nolimits_{\alpha\in\Delta}\hat{g}_\alpha^2 V(\alpha\cdot q), \quad \hat{g}_\alpha^2 = \begin{cases} g_s^2 + \alpha_s^2\tilde{g}_s^2 & \alpha\in\Delta_s \\ g_l^2 + \alpha_l^2\tilde{g}_l^2 & \alpha\in\Delta_l \end{cases}$$

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- From real fields to complex particle systems

  i) No restrictions
  e.g. Benjamin-Ono equation

$$u_t + uu_x + \lambda H u_{xx} = 0 \qquad (*)$$

$H \equiv$ Hilbert transform, i.e. $Hu(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{u(x)}{z-x} dz$
Then

$$u(x,t) = \frac{\lambda}{2} \sum_{k=1}^{\ell} \left(\frac{i}{x - z_k} - \frac{i}{x - z_k^*} \right) \in \mathbb{R}$$

satisfies (*) iff z_k obeys the A_n-Calogero equ. of motion

$$\ddot{z}_k = \frac{\lambda^2}{2} \sum_{k \neq j} (z_j - z_k)^{-3}$$

[H. Chen, N. Pereira, Phys. Fluids 22 (1979) 187]
[talk by J. Feinberg, PHHQP workshop VI, 2007, London]

ii) restrict to submanifold

Theorem: [Airault, McKean, Moser, CPAM, (1977) 95]
Given a Hamiltonian $H(x_1,\ldots,x_n,\dot{x}_1,\ldots,\dot{x}_n)$ with flow

$$x_i = \partial H/\partial \dot{x}_i \qquad \text{and} \qquad \ddot{x}_i = -\partial H/\partial x_i \qquad i = 1,\ldots,n$$

and conserved charges I_j in involution with H,i.e. $\{I_j, H\} = 0$. Then the locus of $\operatorname{grad} I = 0$ is invariant.

Example: Boussinesq equation

$$v_{tt} = a(v^2)_{xx} + bv_{xxxx} + v_{xx} \qquad (**)$$

Then

$$v(x,t) = c\sum\nolimits_{k=1}^{\ell} (x - z_k)^{-2}$$

satisfies (**) iff b=1/12, c=-a/2 and z_k obeys

$$\begin{aligned} \ddot{z}_k &= 2\sum\nolimits_{j\neq k}(z_j - z_k)^{-3} &\Leftrightarrow\quad& \ddot{z}_k = -\frac{\partial H}{\partial z_i} \\ \dot{z}_k &= 1 - \sum\nolimits_{j\neq k}(z_j - z_k)^{-2} &\Leftrightarrow\quad& \operatorname{grad}(I_3 - I_1) = 0 \end{aligned}$$

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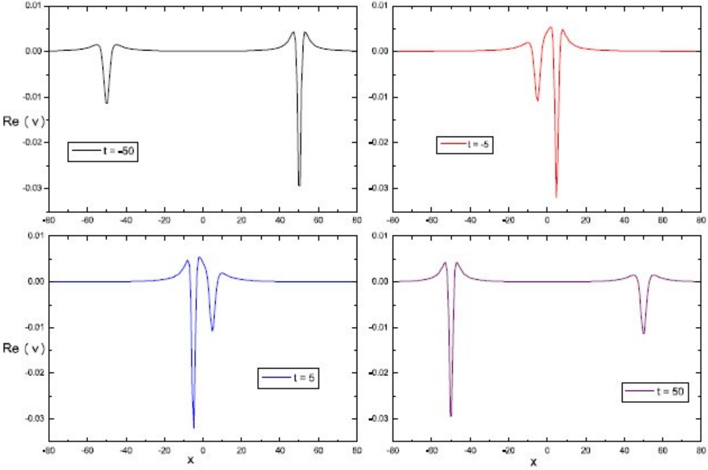
$$\dot{z}_k = 1 - \sum\nolimits_{j\neq k}(z_j - z_k)^{-2} \qquad \Leftrightarrow \qquad \mathrm{grad}(I_3 - I_1) = 0$$

Constrained field equations ⟶ complex Calogero models



[P. Assis and A.F., J. Phys. A42 (2009) 425206]

Consider

Antilinearly invariant deformed Calogero model

$$\mathcal{H}_{\mathcal{PT}\mathrm{CMS}} = \frac{p^2}{2} + \frac{m^2}{16} \sum_{\alpha \in \Delta_s} (\alpha \cdot \tilde{q})^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha V(\alpha \cdot \tilde{q}), \;\; m, g_\alpha \in \mathbb{R}$$

Define deformed coordinates (A_2)

$$
\begin{aligned}
q_1 &\rightarrow \tilde{q}_1 = q_1 \cosh\varepsilon + i\sqrt{3}(q_2 - q_3)\sinh\varepsilon \\
q_2 &\rightarrow \tilde{q}_2 = q_2 \cosh\varepsilon + i\sqrt{3}(q_3 - q_1)\sinh\varepsilon \\
q_3 &\rightarrow \tilde{q}_3 = q_3 \cosh\varepsilon + i\sqrt{3}(q_1 - q_2)\sinh\varepsilon
\end{aligned}
$$

With standard 3D representation for the simple A_2-roots $\alpha_1 = \{1, -1, 0\}$, $\alpha_2 = \{0, 1, -1\}$, $q_{ij} := q_i - q_j$ compute

$$
\begin{aligned}
\alpha_1 \cdot \tilde{q} &= q_{12}\cosh\varepsilon - \frac{\imath}{\sqrt{3}}(q_{13} + q_{23})\sinh\varepsilon, \\
\alpha_2 \cdot \tilde{q} &= q_{23}\cosh\varepsilon - \frac{\imath}{\sqrt{3}}(q_{21} + q_{31})\sinh\varepsilon, \\
(\alpha_1 + \alpha_2) \cdot \tilde{q} &= q_{13}\cosh\varepsilon + \frac{\imath}{\sqrt{3}}(q_{12} + q_{32})\sinh\varepsilon.
\end{aligned}
$$

Symmetries:

$$
\begin{aligned}
\mathcal{S}_1 &: \quad q_1 \leftrightarrow q_2,\ q_3 \leftrightarrow q_3,\ \imath \rightarrow -\imath, \\
\mathcal{S}_2 &: \quad q_2 \leftrightarrow q_3,\ q_1 \leftrightarrow q_1,\ \imath \rightarrow -\imath.
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\end{aligned}
$$

Note, this Hamiltonian also results from deforming the roots:

$$\alpha_1 \quad \rightarrow \quad \tilde{\alpha}_1 = \alpha_1 \cosh \varepsilon \; + i\sqrt{3} \sinh \varepsilon \lambda_2$$
$$\alpha_2 \quad \rightarrow \quad \tilde{\alpha}_2 = \alpha_2 \cosh \varepsilon \; - i\sqrt{3} \sinh \varepsilon \lambda_1$$

Thus

$$\mathcal{H}_{\mathcal{PT}\mathrm{CMS}} = \frac{p^2}{2} + \frac{m^2}{16} \sum_{\tilde{\alpha} \in \tilde{\Delta}_s} (\tilde{\alpha} \cdot q)^2 + \frac{1}{2} \sum_{\tilde{\alpha} \in \tilde{\Delta}} g_{\tilde{\alpha}} V(\tilde{\alpha} \cdot q), \;\; m, g_{\tilde{\alpha}} \in \mathbb{R}$$
$$= \frac{p^2}{2} + \frac{m^2}{16} \sum_{\alpha \in \Delta_s} (\alpha \cdot \tilde{q})^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_{\alpha} V(\alpha \cdot \tilde{q}), \;\; m, g_{\alpha} \in \mathbb{R}$$

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$$\sigma_1^{\varepsilon} \; : \; \tilde{\alpha}_1 \leftrightarrow -\tilde{\alpha}_1, \tilde{\alpha}_2 \leftrightarrow \tilde{\alpha}_1 + \tilde{\alpha}_2 \quad \Leftrightarrow \quad q_1 \leftrightarrow q_2, q_3 \leftrightarrow q_3, \imath \rightarrow -\imath$$
$$\sigma_2^{\varepsilon} \; : \; \tilde{\alpha}_2 \leftrightarrow -\tilde{\alpha}_2, \tilde{\alpha}_1 \leftrightarrow \tilde{\alpha}_1 + \tilde{\alpha}_2 \quad \Leftrightarrow \quad q_2 \leftrightarrow q_3, q_1 \leftrightarrow q_1, \imath \rightarrow -\imath$$

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Construction of antilinear deformations

- Involution $\in \mathcal{W} \equiv$ Coxeter group $\Rightarrow$ deform in antilinear way
- Find a linear deformation map:

$$\delta: \ \Delta \rightarrow \tilde{\Delta}(\varepsilon) \qquad \alpha \mapsto \tilde{\alpha} = \theta_\varepsilon \alpha$$

$$\alpha_i \in \Delta \subset \mathbb{R}^n, \quad \tilde{\alpha}_i(\varepsilon) \in \tilde{\Delta}(\varepsilon) \subset \mathbb{R}^n \oplus \imath\mathbb{R}^n, \quad \varepsilon \in \mathbb{R}$$

- Find a second map that leaves $\tilde{\Delta}(\varepsilon)$ invariant

$$\varpi : \tilde{\Delta}(\varepsilon) \rightarrow \tilde{\Delta}(\varepsilon), \qquad \tilde{\alpha} \mapsto \omega\tilde{\alpha}$$

(i) $\varpi : \tilde{\alpha} = \mu_1\alpha_1 + \mu_2\alpha_2 \mapsto \mu_1^*\omega\alpha_1 + \mu_2^*\omega\alpha_2$ for $\mu_1, \mu_2 \in \mathbb{C}$
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Make the following assumptions

(i) ω decomposes as

$$\omega = \tau\hat{\omega} = \hat{\omega}\tau$$

with $\hat{\omega} \in \mathcal{W}$, $\hat{\omega}^2 = \mathbb{I}$ and complex conjugation τ

(ii) there are at least two different ω_i with $i = 1, 2, \ldots$

(iii) there is a similarity transformation

$$\omega_i := \theta_\varepsilon \hat{\omega}_i \theta_\varepsilon^{-1} = \tau\hat{\omega}_i, \qquad \text{for } i = 1, \ldots, \kappa \geq 2$$

(iv) θ_ε is an isometry for the inner products on $\tilde{\Delta}(\varepsilon)$ therefore

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Many solutions were constructed

$\tilde{\Delta}(\varepsilon)$ **for** A_3

$$\theta_\varepsilon = r_0 \mathbb{I} + r_2 \sigma^2 + \imath r_1 \left(\sigma - \sigma^3 \right)$$

with explicit representation

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \sigma = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix},$$

$\sigma_- = \sigma_1 \sigma_3$, $\sigma_+ = \sigma_2$, $\sigma = \sigma_- \sigma_+$

$$\theta_\varepsilon = \begin{pmatrix} r_0 - \imath r_1 & -2\imath r_1 & -\imath r_1 - r_2 \\ 2\imath r_1 & r_0 - r_2 + 2\imath r_1 & 2\imath r_1 \\ -\imath r_1 - r_2 & -2\imath r_1 & r_0 - \imath r_1 \end{pmatrix}$$

all constraints require

$$\begin{aligned}(r_0+r_2)\left[(r_0+r_2)^2-4r_1^2\right] &= 1\\ r_0-r_2+2r_1 &= (r_0-r_2+2r_1)(r_0+r_2)\\ (r_0+r_2) &= (r_0-r_2)^2-4r_1^2\end{aligned}$$

these are solved by

$$r_0(\varepsilon)=\cosh\varepsilon,\quad r_1(\varepsilon)=\pm\sqrt{\cosh^2\varepsilon-\cosh\varepsilon},\quad r_2(\varepsilon)=1-\cosh\varepsilon$$

$\Rightarrow$ simple deformed roots

$$\tilde{\alpha}_1=\cosh\varepsilon\alpha_1+(\cosh\varepsilon-1)\alpha_3-\imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1+2\alpha_2+\alpha_3),$$

$$\tilde{\alpha}_2=(2\cosh\varepsilon-1)\alpha_2+2\imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1+\alpha_2+\alpha_3),$$

$$\tilde{\alpha}_3=\cosh\varepsilon\alpha_3+(\cosh\varepsilon-1)\alpha_1-\imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1+2\alpha_2+\alpha_3).$$

remaining positive roots

$\tilde{\alpha}_4:=\tilde{\alpha}_1+\tilde{\alpha}_2$, $\tilde{\alpha}_5:=\tilde{\alpha}_2+\tilde{\alpha}_3$, $\tilde{\alpha}_6:=\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3$.

$\tilde{\Delta}(\varepsilon)$ for A_{4n-1}-subseries

closed solution

$$\theta_\varepsilon = r_0 \mathbb{I} + r_{2n}\sigma^{2n} + \imath r_n \left(\sigma^n - \sigma^{-n}\right),$$

- with $r_{2n} = 1 - r_0$, $r_n = \pm\sqrt{r_0^2 - r_0}$
- useful choice $r_0 = \cosh\varepsilon$

$\tilde{\Delta}(\varepsilon)$ for E_6

$$\theta_\varepsilon = \begin{pmatrix} r_0 & -2\imath r_2 & 0 & -2\imath r_2 & -2\imath r_2 & -\imath r_2 \\ 2\imath r_2 & r_0 + \imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 \\ 0 & 2\imath r_2 & r_0 + 2\imath r_2 & 4\imath r_2 & 3\imath r_2 & 2\imath r_2 \\ -2\imath r_2 & -2\imath r_2 & -4\imath r_2 & r_0 - 5\imath r_2 & -4\imath r_2 & -2\imath r_2 \\ 2\imath r_2 & 2\imath r_2 & 3\imath r_2 & 4\imath r_2 & r_0 + 2\imath r_2 & 0 \\ -\imath r_2 & -2\imath r_2 & -2\imath r_2 & -2\imath r_2 & 0 & r_0 \end{pmatrix}$$

$r_2 = \pm 1/\sqrt{3}\sqrt{r_0^2 - 1}$, $r_0 = \cosh\varepsilon$

$\tilde{\Delta}(\varepsilon)$ for B_{2n+1}-subseries

no solution based on factorisation of the Coxeter element

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$\tilde{\Delta}(\varepsilon)$ for B_{2n+1}-subseries

no solution based on factorisation of the Coxeter element

with different ω_i we find for instance for B_{2n+1}

$$\begin{aligned}
\tilde{\alpha}_{2j-1} &= \cosh\varepsilon\alpha_{2j-1} + i\sinh\varepsilon\left(\alpha_{2j-1} + 2\sum_{k=2j}^{\ell}\alpha_k\right) \qquad \text{for } j = 1,\ldots, \\
\tilde{\alpha}_{2j} &= \cosh\varepsilon\alpha_{2j} - i\sinh\varepsilon\left(\sum_{k=2j}^{2j+2}\alpha_k + 2\sum_{k=2j+3}^{\ell}2\alpha_k\right) \quad \text{for } j = 1,\ldots \\
\tilde{\alpha}_{\ell-1} &= \cosh\varepsilon(\alpha_{\ell-1}+\alpha_\ell) - \alpha_\ell - i\sinh\varepsilon\left(\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell\right), \\
\tilde{\alpha}_\ell &= \alpha_\ell.
\end{aligned}$$

in dual space

$$\theta_\varepsilon^\star = \begin{pmatrix} R & & & & \\ & R & & 0 & \\ & & R & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad \text{with } R = \begin{pmatrix} \cosh\varepsilon & i\sinh\varepsilon \\ -i\sinh\varepsilon & \cosh\varepsilon \end{pmatrix}$$

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$$

For **any** model based on roots, these deformed roots can be used to define new invariant models simply by

$$\alpha \to \tilde{\alpha}.$$

For instance Calogero models:

- Physical properties (A_2, G_2)
  - The deformed model can be solved by separation of variables as the undeformed case.
  - Some restrictions cease to exist, as the wavefunctions are now regularized.
  - $\Rightarrow$ modified energy spectrum:

$$E = 2\left|\omega\right|\left(2n + \lambda + 1\right)$$

becomes

$$E_{n\ell}^{\pm} = 2|\omega|\left[2n + 6(\kappa_s^{\pm} + \kappa_l^{\pm} + \ell) + 1\right] \qquad \text{for } n, \ell \in \mathbb{N}_0,$$

with $\kappa_{s/l}^{\pm} = (1 \pm \sqrt{1 + 4g_{s/l}})/4$

[A. Fring and M. Znojil, J. Phys. A41 (2008) 194010]

- generalized Calogero Hamiltonian (undeformed)

$$\mathcal{H}_C(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_{\alpha\in\Delta^+}(\alpha\cdot q)^2 + \sum_{\alpha\in\Delta^+}\frac{g_\alpha}{(\alpha\cdot q)^2},$$

- define the variables

$$z := \prod_{\alpha\in\Delta^+}(\alpha\cdot q) \qquad \text{and} \qquad r^2 := \frac{1}{\hat{h}t_\ell}\sum_{\alpha\in\Delta^+}(\alpha\cdot q)^2,$$

$\hat{h} \equiv$ dual Coxeter number, $t_\ell \equiv \ell$-th symmetrizer of I

- Ansatz:

$$\psi(q) \to \psi(z,r) = z^{\kappa+1/2}\varphi(r)$$

$\Rightarrow$ solution for $\kappa = 1/2\sqrt{1+4g}$.

$$\varphi_n(r) = c_n \exp\left(-\sqrt{\frac{\hat{h}t_\ell}{2}}\frac{\omega}{2}r^2\right) L_n^a\left(\sqrt{\frac{\hat{h}t_\ell}{2}}\omega r^2\right).$$

$L_n^a(x) \equiv$ Laguerre polynomial, $a = \left(2+h+h\sqrt{1+4g}\right)\ell/4 - 1$

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• eigenenergies

$$E_n = \frac{1}{4}\left[\left(2+h+h\sqrt{1+4g}\right)l+8n\right]\sqrt{\frac{\hat{h}t_\ell}{2}}\omega$$

• anyonic exchange factors

$$\psi(q_1,\ldots,q_i,q_j,\ldots q_n) = e^{\imath\pi s}\psi(q_1,\ldots,q_j,q_i,\ldots q_n), \quad \text{for } 1 \leq i,j \leq n,$$

with

$$s = \frac{1}{2}+\frac{1}{2}\sqrt{1+4g}$$

$\because$ r is symmetric and z antisymmetric

The construction is based on the identities:

$$
\begin{aligned}
\sum_{\alpha,\beta\in\Delta^+} \frac{\alpha\cdot\beta}{(\alpha\cdot q)(\beta\cdot q)} &= \sum_{\alpha\in\Delta^+} \frac{\alpha^2}{(\alpha\cdot q)^2}, \\
\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\frac{(\alpha\cdot q)}{(\beta\cdot q)} &= \frac{\hat{h}h\ell}{2} t_\ell, \\
\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\,(\alpha\cdot q)(\beta\cdot q) &= \hat{h}t_\ell \sum_{\alpha\in\Delta^+} (\alpha\cdot q)^2, \\
\sum_{\alpha\in\Delta^+} \alpha^2 &= \ell\hat{h}t_\ell .
\end{aligned}
$$

Strong evidence on a case-by-case level, but no rigorous proof.

- antilinearly deformed Calogero Hamiltionian

$$\mathcal{H}_{adC}(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4} \sum_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q)^2 + \sum_{\tilde{\alpha}\in\Delta^+} \frac{g_{\tilde{\alpha}}}{(\tilde{\alpha}\cdot q)^2}$$

- define the variables

$$\tilde{z} := \prod_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q) \qquad \text{and} \qquad \tilde{r}^2 := \frac{1}{\hat{h}t_\ell} \sum_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q)^2$$

- Ansatz

$$\psi(q) \to \psi(\tilde{z},\tilde{r}) = \tilde{z}^s \varphi(\tilde{r})$$

when identies still hold ⇒

$$\psi(q) = \psi(\tilde{z},r) = \tilde{z}^s \varphi_n(r)$$

eigenenergies with different constraints (only performed for ground state)

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Deformed A_3-models

- potential from deformed Coxeter group factors
  $\alpha_1 = \{1, -1, 0, 0\}$, $\alpha_2 = \{0, 1, -1, 0\}$, $\alpha_3 = \{0, 0, 1, -1\}$

$$
\begin{aligned}
\tilde{\alpha}_1 \cdot q &= q_{43} + \cosh\varepsilon(q_{12} + q_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{13} + q_{24}) \\
\tilde{\alpha}_2 \cdot q &= q_{23}(2\cosh\varepsilon - 1) + \imath 2\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2} q_{14} \\
\tilde{\alpha}_3 \cdot q &= q_{21} + \cosh\varepsilon(q_{12} + q_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{13} + q_{24}) \\
\tilde{\alpha}_4 \cdot q &= q_{42} + \cosh\varepsilon(q_{13} + q_{24}) + \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{12} + q_{34}) \\
\tilde{\alpha}_5 \cdot q &= q_{31} + \cosh\varepsilon(q_{13} + q_{24}) + \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{12} + q_{34}) \\
\tilde{\alpha}_6 \cdot q &= q_{14}(2\cosh\varepsilon - 1) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2} q_{23}
\end{aligned}
$$

notation $q_{ij} = q_i - q_j$, **No longer singular for $q_{ij} = 0$**

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- $\mathcal{PT}$-symmetry for $\tilde{\alpha}$

$$\sigma_-^\varepsilon : \tilde{\alpha}_1 \to -\tilde{\alpha}_1, \tilde{\alpha}_2 \to \tilde{\alpha}_6, \tilde{\alpha}_3 \to -\tilde{\alpha}_3, \tilde{\alpha}_4 \to \tilde{\alpha}_5, \tilde{\alpha}_5 \to \tilde{\alpha}_4, \tilde{\alpha}_6 \to \tilde{\alpha}_2$$
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- $\mathcal{PT}$-symmetry in dual space

$$\sigma_-^\varepsilon : q_1 \to q_2, q_2 \to q_1, q_3 \to q_4, q_4 \to q_3, \imath \to -\imath$$
$$\sigma_+^\varepsilon : q_1 \to q_1, q_2 \to q_3, q_3 \to q_2, q_4 \to q_4, \imath \to -\imath$$

$\Rightarrow$

$$\begin{aligned}
\sigma_-^\varepsilon \tilde{z}(q_1, q_2, q_3, q_4) &= \tilde{z}^*(q_2, q_1, q_4, q_3) = \tilde{z}(q_1, q_2, q_3, q_4) \\
\sigma_+^\varepsilon \tilde{z}(q_1, q_2, q_3, q_4) &= \tilde{z}^*(q_1, q_3, q_2, q_4) = -\tilde{z}(q_1, q_2, q_3, q_4)
\end{aligned}$$

$$\psi(q_1, q_2, q_3, q_4) = e^{\imath \pi s} \psi(q_2, q_4, q_1, q_3).$$

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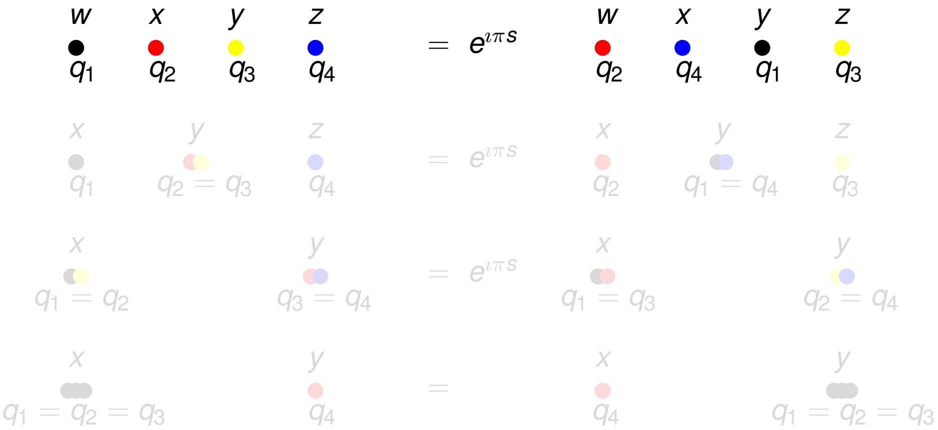
Anyonic exchange factors in the 4-particle scattering process



Anyonic exchange factors in the 4-particle scattering process

$$\begin{array}{cccc} w & x & y & z \\ q_1 & q_2 & q_3 & q_4 \end{array} = e^{\imath\pi s} \begin{array}{cccc} w & x & y & z \\ q_2 & q_4 & q_1 & q_3 \end{array}$$

$$\begin{array}{ccc} x & y & z \\ q_1 & q_2 = q_3 & q_4 \end{array} = e^{\imath\pi s} \begin{array}{ccc} x & y & z \\ q_2 & q_1 = q_4 & q_3 \end{array}$$

$$\begin{array}{cc} x & y \\ q_1 = q_2 & q_3 = q_4 \end{array} = e^{\imath\pi s} \begin{array}{cc} x & y \\ q_1 = q_3 & q_2 = q_4 \end{array}$$

$$\begin{array}{cc} x & y \\ q_1 = q_2 = q_3 & q_4 \end{array} = \begin{array}{cc} x & y \\ q_4 & q_1 = q_2 = q_3 \end{array}$$

Anyonic exchange factors in the 4-particle scattering process

$$\begin{array}{cccc} w & x & y & z \\ q_1 & q_2 & q_3 & q_4 \end{array} \quad = \quad e^{\imath\pi s} \quad \begin{array}{cccc} w & x & y & z \\ q_2 & q_4 & q_1 & q_3 \end{array}$$

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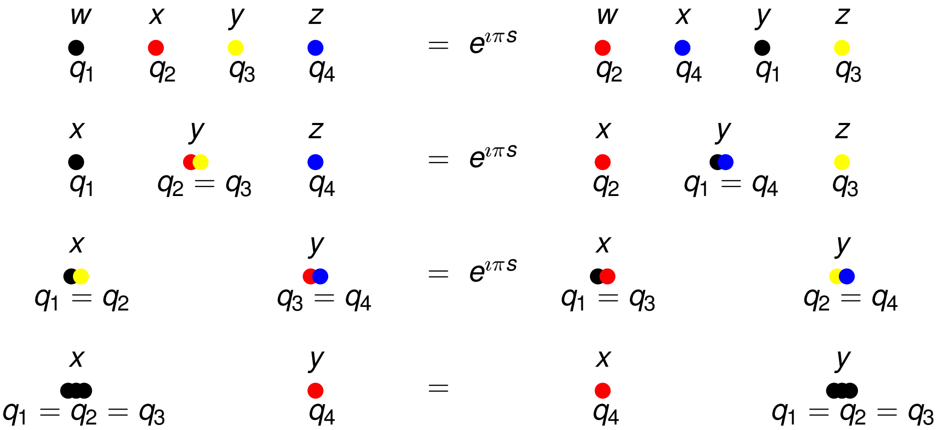

Find Hermitian counterpart h, Dyson map η and metric ρ:

$$h = \eta H \eta^{-1} = h^{\dagger} = (\eta^{-1})^{\dagger} H^{\dagger} \eta^{\dagger} \quad \Leftrightarrow \quad H^{\dagger}\rho = \rho H \;\text{ with } \rho = \eta^{\dagger}\eta$$

Some B_{ℓ}-models correspond to complex rotations

$$\begin{pmatrix} \tilde{z}_i \\ \tilde{z}_j \end{pmatrix} = R_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} = \eta_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} \eta_{ij}^{-1}, \qquad \text{for } z \in \{x, p\},\ \eta_{ij} = e^{\varepsilon(x_i p_j - x_j p_i)}$$

For instance for:

$$\theta_{\varepsilon}^{\star} = \begin{pmatrix} R & & & & \\ & R & & 0 & \\ & & R & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad \text{with } R = \begin{pmatrix} \cosh\varepsilon & i\sinh\varepsilon \\ -i\sinh\varepsilon & \cosh\varepsilon \end{pmatrix}$$

we have

$$\mathcal{H}_0(p, x) = \eta \mathcal{H}_{\varepsilon}(p, x) \eta^{-1}$$

with

$$\eta = \eta_{12}^{-1} \eta_{34}^{-1} \eta_{56}^{-1} \cdots \eta_{(\ell-2)(\ell-1)}^{-1}$$

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$$\theta_\varepsilon^\star = \begin{pmatrix} R & & & \\ & R & 0 & \\ & & R & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad \text{with } R = \begin{pmatrix} \cosh\varepsilon & i\sinh\varepsilon \\ -i\sinh\varepsilon & \cosh\varepsilon \end{pmatrix}$$

we have

$$\mathcal{H}_0(p, x) = \eta \mathcal{H}_\varepsilon(p, x) \eta^{-1}$$

with

$$\eta = \eta_{12}^{-1} \eta_{34}^{-1} \eta_{56}^{-1} \cdots \eta_{(\ell-2)(\ell-1)}^{-1}$$

Find Hermitian counterpart h, Dyson map η and metric ρ:

$$h = \eta H \eta^{-1} = h^{\dagger} = (\eta^{-1})^{\dagger} H^{\dagger} \eta^{\dagger} \quad \Leftrightarrow \quad H^{\dagger}\rho = \rho H \;\text{ with } \rho = \eta^{\dagger}\eta$$

Some B_ℓ-models correspond to complex rotations

$$\begin{pmatrix} \tilde{z}_i \\ \tilde{z}_j \end{pmatrix} = R_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} = \eta_{ij} \begin{pmatrix} z_i \\ z_j \end{pmatrix} \eta_{ij}^{-1}, \qquad \text{for } z \in \{x, p\},\ \eta_{ij} = e^{\varepsilon(x_i p_j - x_j p_i)}$$

For instance for:

$$\theta_\varepsilon^\star = \begin{pmatrix} R & & & & \\ & R & & 0 & \\ & & R & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad \text{with } R = \begin{pmatrix} \cosh\varepsilon & i\sinh\varepsilon \\ -i\sinh\varepsilon & \cosh\varepsilon \end{pmatrix}$$

we have

$$\mathcal{H}_0(p, x) = \eta \mathcal{H}_\varepsilon(p, x) \eta^{-1}$$

with

$$\eta = \eta_{12}^{-1} \eta_{34}^{-1} \eta_{56}^{-1} \ldots \eta_{(\ell-2)(\ell-1)}^{-1}$$

For B_5

$$\theta_\varepsilon^\star = \begin{pmatrix} r_0 & -i\vartheta & i\vartheta & 1-r_0 & 0 \\ i\vartheta & r_0 & 1-r_0 & -i\vartheta & 0 \\ -i\vartheta & 1-r_0 & r_0 & i\vartheta & 0 \\ 1-r_0 & i\vartheta & -i\vartheta & r_0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

we find

$$\tilde{x} = \theta_\varepsilon^\star x = R_{24}^{-1} R_{13} R_{34} R_{12}^{-1} x = \eta x \eta^{-1}, \qquad \text{with } \eta = \eta_{24}^{-1} \eta_{13} \eta_{34} \eta_{12}^{-1}.$$

In general this is an open problem.

General deformation prescription:

$\mathcal{PT}$-anti-symmetric quantities:

$$\mathcal{PT} : \phi(x,t) \mapsto -\phi(x,t) \quad \Rightarrow \quad \delta_\varepsilon : \phi(x,t) \mapsto -i[i\phi(x,t)]^\varepsilon$$

Two possibilities for the KdV Hamiltonian

$$\delta_\varepsilon^+ : u_x \mapsto u_{x,\varepsilon} := -i(iu_x)^\varepsilon \qquad \text{or} \qquad \delta_\varepsilon^- : u \mapsto u_\varepsilon := -i(iu)^\varepsilon,$$

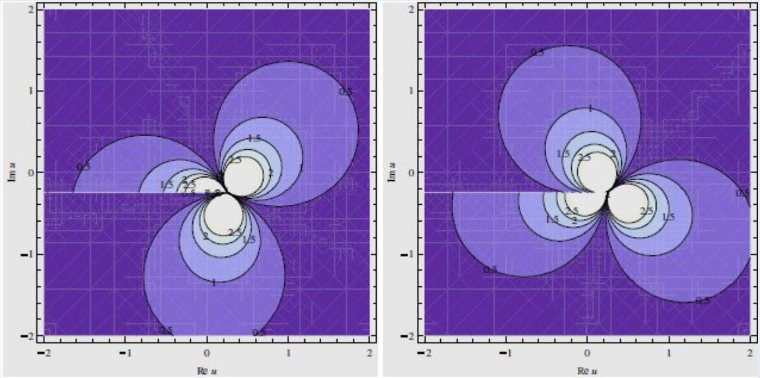
such that

$$\mathcal{H}_\varepsilon^+ = -\frac{\beta}{6}u^3 - \frac{\gamma}{1+\varepsilon}(iu_x)^{\varepsilon+1} \qquad \mathcal{H}_\varepsilon^- = \frac{\beta}{(1+\varepsilon)(2+\varepsilon)}(iu)^{\varepsilon+2} + \frac{\gamma}{2}u_x^2$$

with equations of motion

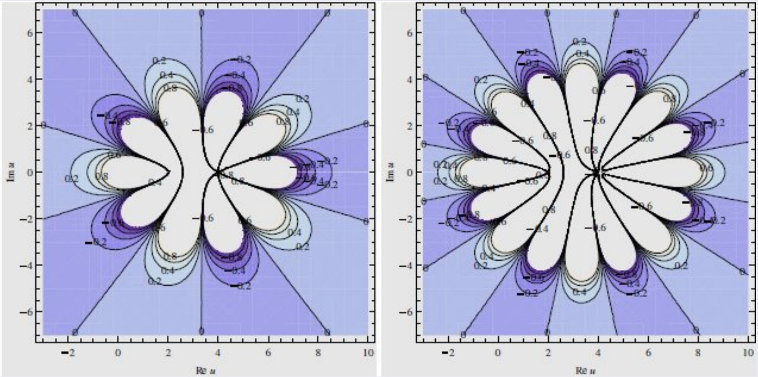
$$u_t + \beta u u_x + \gamma u_{xxx,\varepsilon} = 0 \qquad u_t + i\beta u_\varepsilon u_x + \gamma u_{xxx} = 0$$

Broken $\mathcal{PT}$-symmetric rational solutions for $\mathcal{H}_{1/3}^{+}$



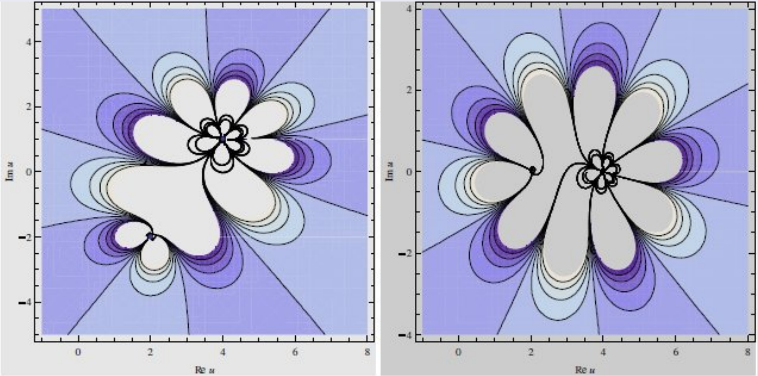
Different Riemann sheets for $A = (1-i)/4$, $c = 1$, $\beta = 2 + 2i$ and $\gamma = 3$
(a) $u^{(1)}$
(b) $u^{(2)}$

$\mathcal{PT}$-symmetric trigonometric/hyperbolic solutions



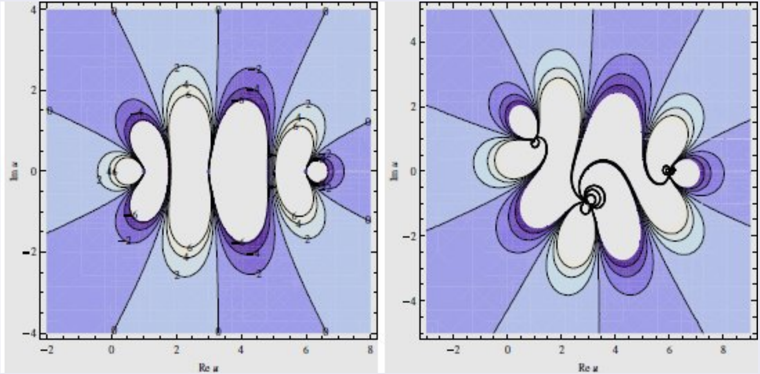
$A = 4, B = 2, c = 1, \beta = 2$ and $\gamma = 3$
(a) $\mathcal{H}^+_{-1/2}$
(b) $\mathcal{H}^+_{-2/3}$

Broken $\mathcal{PT}$-symmetric trigonometric solutions for $\mathcal{H}_{-1/2}^{+}$



(a) Spontaneously broken $\mathcal{PT}$-symmetry with $A = 4 + i$, $B = 2 - 2i$, $c = 1$, $\beta = 3/10$ and $\gamma = 3$
(b) broken $\mathcal{PT}$-symmetry with $A = 4$, $B = 2$, $c = 1$, $\beta = 3/10$ and $\gamma = 3 + i$

Elliptic solutions for $\mathcal{H}_{-1/2}^{+}$:



(a) $\mathcal{PT}$-symmetric with $A = 1$, $B = 3$, $C = 6$, $\beta = 3/10$, $\gamma = -3$ and $c = 1$
(b) spontaneously broken $\mathcal{PT}$-symmetry with $A = 1 + i$, $B = 3 - i$, $C = 6$, $\beta = 3/10$, $\gamma = -3$ and $c = 1$

The $\mathcal{H}_\varepsilon^-$-models

Integrating twice gives now:

$$u_\zeta^2 = \frac{2}{\gamma}\left(\kappa_2 + \kappa_1 u + \frac{c}{2}u^2 - \beta\frac{i^\varepsilon}{(1+\varepsilon)(2+\varepsilon)}u^{2+\varepsilon}\right) =: \lambda Q(u)$$

where

$$\lambda = -\frac{2\beta i^\varepsilon}{\gamma(1+\varepsilon)(2+\varepsilon)}$$

For $\kappa_1 = \kappa_2 = 0$

$$u(\zeta) = \left(\frac{c(\varepsilon+1)(\varepsilon+2)}{i^\varepsilon\beta\left[\cosh\left(\frac{\sqrt{c}\varepsilon(\zeta-\zeta_0)}{\sqrt{\gamma}}\right)+1\right]}\right)^{1/\varepsilon}$$

- $\mathcal{H}_2^-$:
  $\equiv$ complex version of the modified KdV-equation
- $\mathcal{H}_4^-$:
  assume $Q(u) = u^2(u^2 - B^2)(u^2 - C^2)$, possible for

$$\kappa_1 = \kappa_2 = 0, \qquad B = iC \qquad \text{and} \qquad C^4 = \frac{15c}{\beta}$$

eigenvalues of Jacobian:

$$j_1 = \pm i\sqrt{r_\lambda} r_B^2 \exp\left[\frac{i}{2}(4\theta_B + \theta_\lambda)\right]$$

$$j_2 = \mp i\sqrt{r_\lambda} r_B^2 \exp\left[-\frac{i}{2}(4\theta_B + \theta_\lambda)\right]$$

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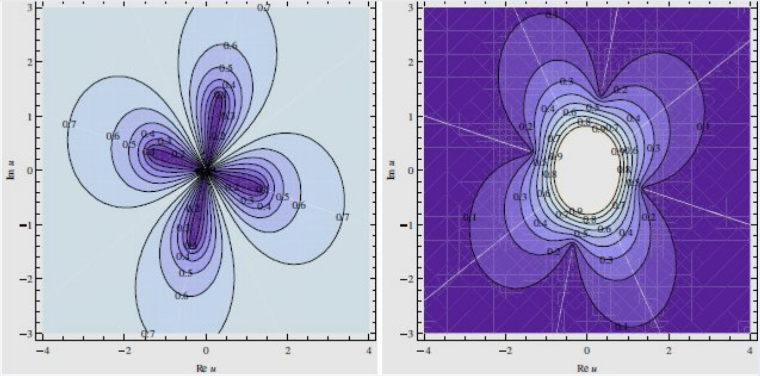
Broken $\mathcal{PT}$-symmetric solution for $\mathcal{H}_4^-$:



(a) star node at the origin for $c = 1$, $\beta = 2 + i3$, $\gamma = 1$ and $B = (15/2 + i3)^{1/4}$
(b) centre at the origin for $c = 1$, $\beta = 2 + i3$, $\gamma = -1$ and $B = (30/13 - i45/13)^{1/4}$

Reduction to quantum mechanical Hamiltonians:

Again we can relate to simple quantum mechanical models:
The identification

$$u \to x, \quad \zeta \to t, \quad \kappa_1 = 0, \quad \kappa_2 = \gamma E, \quad \text{and} \quad \beta = \gamma g(1+\varepsilon)(2+\varepsilon)$$

relates $\mathcal{H}_{\varepsilon}^{-}$ to

$$H = E = \frac{1}{2}p^2 - \frac{c}{2\gamma}x^2 + gx^2(ix)^{\varepsilon}$$

For $c = 0$ these are the "classical models" studied in
[C. Bender, S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243]

Reduction of the $\mathcal{H}_2^-$-model

$$\mathcal{H}_2^-[u] = \frac{\beta}{12}u^4 + \frac{\gamma}{2}u_x^2$$

Twice integrated equation of motion:

$$u_\zeta^2 = \frac{2}{\gamma}\left(\kappa_2 + \kappa_1 u + \frac{c}{2}u^2 + \beta\frac{1}{12}u^4\right) =: \lambda Q(u)$$

Reduction $u \to x, \zeta \to t$

$$\kappa_1 = -\gamma\tau, \quad \kappa_2 = \gamma E_x, \quad \beta = -3\gamma g \quad \text{and} \quad c = -\gamma\omega^2$$

Quartic harmonic oscillator of the form

$$H = E_x = \frac{1}{2}p^2 + \tau x + \frac{\omega^2}{2}x^2 + \frac{g}{4}x^4$$

Boundary cond.: $\kappa_1 = \tau = 0, \lim_{\zeta\to\infty} u(\zeta) = 0, \lim_{\zeta\to\infty} u_x(\zeta) = \sqrt{2E_x}$

[A.G. Anderson, C. Bender, U. Morone, arXiv:1102.4822]

Note: $E_x \neq E_u(a)$

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Ito type systems and its deformations

Coupled nonlinear system

$$\begin{aligned} u_t + \alpha v v_x + \beta u u_x + \gamma u_{xxx} &= 0, \qquad \alpha, \beta, \gamma \in \mathbb{C}, \\ v_t + \delta (uv)_x + \phi v_{xxx} &= 0, \qquad \delta, \phi \in \mathbb{C} \end{aligned}$$

Hamiltonian for $\delta = \alpha$

$$\mathcal{H}_I = -\frac{\alpha}{2} u v^2 - \frac{\beta}{6} u^3 + \frac{\gamma}{2} u_x^2 + \frac{\phi}{2} v_x^2$$

$\mathcal{PT}$-symmetries:

$$\mathcal{PT}_{++} : x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto u, v \mapsto v \qquad \text{for } \alpha, \beta, \gamma, \phi \in \mathbb{R}$$
$$\mathcal{PT}_{+-} : x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto u, v \mapsto -v \qquad \text{for } \alpha, \beta, \gamma, \phi \in \mathbb{R}$$
$$\mathcal{PT}_{-+} : x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto -u, v \mapsto v \qquad \text{for } i\alpha, i\beta, \gamma, \phi \in \mathbb{R}$$
$$\mathcal{PT}_{--} : x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto -u, v \mapsto -v \qquad \text{for } i\alpha, i\beta, \gamma, \phi \in \mathbb{R}$$

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Deformed models

$$
\begin{aligned}
\mathcal{H}^{++}_{\varepsilon,\mu} &= -\frac{\alpha}{2}uv^2 - \frac{\beta}{6}u^3 - \frac{\gamma}{1+\varepsilon}(iu_x)^{\varepsilon+1} - \frac{\phi}{1+\mu}(iv_x)^{\mu+1} \\
\mathcal{H}^{+-}_{\varepsilon,\mu} &= \frac{\alpha}{1+\mu}u(iv)^{\mu+1} - \frac{\beta}{6}u^3 - \frac{\gamma}{1+\varepsilon}(iu_x)^{\varepsilon+1} + \frac{\phi}{2}v_x^2 \\
\mathcal{H}^{-+}_{\varepsilon,\mu} &= -\frac{\alpha}{2}uv^2 - \frac{i\beta}{(1+\varepsilon)(2+\varepsilon)}(iu)^{2+\varepsilon} + \frac{\gamma}{2}u_x^2 - \frac{\phi}{1+\mu}(iv_x)^{\mu+1} \\
\mathcal{H}^{--}_{\varepsilon,\mu} &= \frac{\alpha}{1+\mu}u(iv)^{\mu+1} - \frac{i\beta}{(1+\varepsilon)(2+\varepsilon)}(iu)^{2+\varepsilon} + \frac{\gamma}{2}u_x^2 + \frac{\phi}{2}v_x^2
\end{aligned}
$$

with equations of motion

$$
\begin{aligned}
u_t + \alpha v v_x + \beta u u_x + \gamma u_{xxx,\varepsilon} &= 0, & u_t + \alpha v_\mu v_x + \beta u u_x + \gamma u_{xxx,\varepsilon} &= 0, \\
v_t + \alpha(uv)_x + \phi v_{xxx,\mu} &= 0, & v_t + \alpha(uv_\mu)_x + \phi v_{xxx} &= 0,
\end{aligned}
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v_t + \alpha(uv)_x + \phi v_{xxx,\mu} &= 0, & v_t + \alpha(uv_\mu)_x + \phi v_{xxx} &= 0.
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Some general conclusions

- Non-Hermitian Hamiltonians describe physical systems within a self-consistent quantum mechanical framework.
- One can use this possibility to explore deformations of well studied models, e.g. integrable systems.
- There exist now experiments in optics for the broken PT-regime.

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Special issue on quantum physics with non-Hermitian operators to be published in Journal of Physics A: Mathematical and Theoretical

guest editors:
Carl Bender, Andreas Fring, Uwe Guenther, Hugh Jones

The deadline for contributed papers will be **31 March 2012.**

Thank you for your attention