HYPERKÄHLER AND QUATERNIONIC KÄHLER GEOMETRY

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THE QK/HK CORRESPONDENCE

A.Haydys, *Hyperkähler and quaternionic Kähler manifolds with S^1 symmetries*, Jour.Geom.Phys. **58** (2008) 293–306.

S.Alexandrov, D.Persson and B.Pioline, *Wall-crossing, Rogers dilogarithm, and the QK/HK correspondence*, arXiv 1110.0466

A.Neitzke *On a hyperholomorphic line bundle over the Coulomb branch*, arXiv 1110.1619

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Then, if $U \subset P$ is the open set on which S^1 acts freely, U/S^1 inherits a quaternionic Kähler structure

- $U(1)$ acts on the quaternionic Kähler manifold U/S^1

- Any quaternionic Kähler manifold with a circle action arises this way.

- 1. THE HYPERHOLOMORPHIC LINE BUNDLE

- 2. THE CORRESPONDENCE

THE HYPERHOLOMORPHIC LINE BUNDLE

- S^1-action on hyperkähler manifold M
- $\mathcal{L}_X\omega_1 = 0 \quad \mathcal{L}_X\omega_2 = \omega_3 \quad \mathcal{L}_X\omega_3 = -\omega_2$
- moment map μ for ω_1: $i_X\omega_1 = d\mu$

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$$(dd_1^c f = dIdf, dd_2^c f = dJdf, dd_3^c f = dKdf)$$

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- $\mathcal{L}_X Y = \nabla_X Y - (\nabla X)(Y)$

- ∇_X acts trivially on each Kähler form, $\mathcal{L}_X$ acts trivially on ω_1

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- $\Lambda_1 dd_1^c \mu = -\Delta\mu$

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- so $\Lambda_1(\omega_1 - dd_1^c\mu) = 0$

(e.g. in $\mathbf{C}^2$, $\omega_1 - dd_1^c\mu = i(dzd\bar{z} - dwd\bar{w})/2$)

- If $[\omega_1]/2\pi$ is an integral class $F = \omega_1 - dd_1^c\mu$ is the curvature of a connection on a principal S^1-bundle

- F type $(1,1)$ with respect to all complex structures

$$= \text{hyperholomorphic}$$

LIFTING THE CIRCLE ACTION

- principal bundle $P \to M$

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- vector field X on M
- $i_X F = df$ = infinitesimal lift $\mathcal{L}_X = \nabla_X + if$
- two lifts differ by fiber action $p \mapsto e^{in\theta} \cdot p$

  $\sim$ choices of f

- $F = \omega_1 - dd_1^c\mu$

- $i_X F = d\mu - i_X dd_1^c \mu = d\mu - \mathcal{L}_X d_1^c \mu + d(i_X I d\mu)$

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- $i_X F = d\mu - i_X dd_1^c\mu = d\mu - \mathcal{L}_X d_1^c\mu + d(i_X I d\mu)$
- $\mathcal{L}_X\mu = 0, \mathcal{L}_X I = 0$
- $i_X F = d(\mu + i_X I d\mu) = d(\mu + \omega_1(X, IX)) = d(\mu + g(X,X))$

EXAMPLES

- the c-map
- monopole moduli spaces
- cotangent bundles

THE C-MAP

- M compact Calabi-Yau threefold, 3-form $\Omega = \Omega_1 + i\Omega_2$

- $[\Omega_1] \in H^3(M, \mathbf{R})$

  flat coordinates $x_1, \ldots, x_{2n}$ of moduli space $\mathcal{M}$

- skew intersection form ω

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- circle action $\Omega \mapsto e^{i\theta}\Omega$ $\qquad \phi = \int_M \Omega_1 \wedge \Omega_2$

- on $\mathcal{M} \times \mathbf{R}^{2n}$ define

$$\omega_1 + i\omega_2 = \sum \omega_{jk} d(x_j + iy_j) \wedge d(x_k + iy_k)$$

$$\omega_3 = 2 \sum \frac{\partial^2 \phi}{\partial x_j \partial x_k} dx_j \wedge dy_k$$

- (indefinite) hyperkähler metric

- S^1-action generated by Hamiltonian vector field of ϕ wrt ω_1

- $dd_1^c\phi = \sum \omega_{jk} dx_j \wedge dx_k$

- $F = \omega_1 - dd_1^c\phi = \sum \omega_{jk} dy_j \wedge dy_k$

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- $\mathbf{R}^{2n}/\Lambda$ = intermediate Jacobian of M

- S^1-bundle = Heisenberg extension

MONOPOLE MODULI SPACES

- $SU(2)$ Bogomolny equations on $\mathbf{R}^3$

- $F = *\nabla\phi$

$$\|\phi\| \sim 1 - \frac{k}{2r} - \frac{Q(x,x)}{4r^5} + \dots$$

- $4k$-dimensional hyperkähler moduli space, $SO(3)$ action rotating $\omega_1, \omega_2, \omega_3$

- S^1-action rotation about direction u

- moment map

rotationally invariant

$$\mu = \frac{4}{(N+1)(N+2)} \frac{\vartheta^{(N+2)}(0)}{\vartheta^{(N)}(0)} - \frac{1}{3} Q(u,u)$$

NJH, *Integrable systems in Riemannian geometry*, in Surveys in Differential Geometry Vol. 4, C.-L. Terng and K. Uhlenbeck, (eds.), International Press, Cambridge, Mass. (1999), 21– 80.

$$\omega_1 = dd_1^c \left(\frac{4}{(N+1)(N+2)} \frac{\vartheta^{(N+2)}(0)}{\vartheta^{(N)}(0)} - \frac{1}{3} Q(v,v) \right)$$

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- $F = \omega_1 - dd^c_1 \mu = -\frac{1}{3} dd^c_1 (Q(u,u) - Q(v,v)) = -\frac{1}{3} dd^c_1 Q(u,u)$

- ($Q(v,v)$ real part of an I-holomorphic function)

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- trivial holomorphic bundle

- Hermitian metric $\exp(-Q(u,u)/3)$

COTANGENT BUNDLES

THM. (Feix, Kaledin). Let M be a real analytic Kähler manifold, then there is a unique S^1-invariant hyperkähler metric on a neighbourhood of the zero section extending the Kähler metric.

B. Feix, *Hyperkähler metrics on cotangent bundles*, J. Reine Angew. Math. **532** (2001), 33-46.

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THM. (Feix). Let A be a connection in a vector bundle over M whose curvature is of type $(1,1)$, then this extends to a unique S^1-invariant hyperholomorphic connection on a neighbourhood of the zero section.

B. Feix, *Hypercomplex manifolds and hyperholomorphic bundles,* Math. Proc. Camb. Phil. Soc. **133** (2002), 443-457

- $F = \omega_1 - dd_1^c \mu$ restricts to ω_1 on the zero section

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- M = moduli space of stable bundles

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- = determinant line of the $\bar{\partial}_A$-operator of the Higgs bundle

THE TWISTOR VIEWPOINT

- holomorphic principal $\mathbf{C}^*$-bundle $P \to Z$
- $TP/\mathbf{C}^*$ bundle on Z: Atiyah algebroid A
- holomorphic extension

$$0 \to \mathcal{O} \to A \to T \to 0$$

- classified by $H^1(Z, T^*)$

- S^1-action, fixes ω_1 and rotates ω_2, ω_3

- $\zeta \in \mathbf{C}P^1$ $\zeta \mapsto e^{i\theta}\zeta$ fixed points $0, \infty$

- fibres of Z over $0, \infty$, Z_0, Z_∞ are preserved

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- (hyperholomorphic curvature $\omega_1 - dd_1^c\mu \Rightarrow$ holomorphically equivalent by e^μ to bundle with curvature ω_1)

- $i_X\omega_1 = d\mu$: lift of action to L

- $\Rightarrow$ holomorphic section $\tilde{X}$ of A

$$0 \to \mathcal{O} \to A \to T \to 0$$

$$\tilde{X} \to X$$

- twistor space $Z \to \mathbf{C}P^1$
- cotangent bundle of Z on Z_0

$$0 \to \mathcal{O}(-2) \to T^*Z \to T^*Z_0 \to 0$$

- defined by $(\omega_2 + i\omega_3)^{-1}\omega_1 \in \Omega^{0,1}(TZ_0(-2))$

- $(\omega_2 + i\omega_3) : TZ_0 \to T^*Z_0(2)$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "="] & A \arrow[r] \arrow[d, "\cong"] & TZ_0 \arrow[r] \arrow[d, "\omega_2+i\omega_3"] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] & T^*Z(2) \arrow[r] & T^*Z_0(2) \arrow[r] & 0
\end{tikzcd}$$

- $\tilde{X} \in A \mapsto \theta_0 \in T^*Z(2)$ on Z_0

- repeat at ∞

- $D = Z_0 + Z_\infty$ zero set of ζ, section of $\mathcal{O}(2)$
- $0 \to T^*Z \xrightarrow{\zeta} T^*Z(2) \to T^*Z(2)|_D \to 0$
- $\theta \in H^0(D, T^*Z(2)) \to H^1(Z, T^*Z)$
- defines the Atiyah algebroid of the hyperholomorphic line bundle

THE CORRESPONDENCE

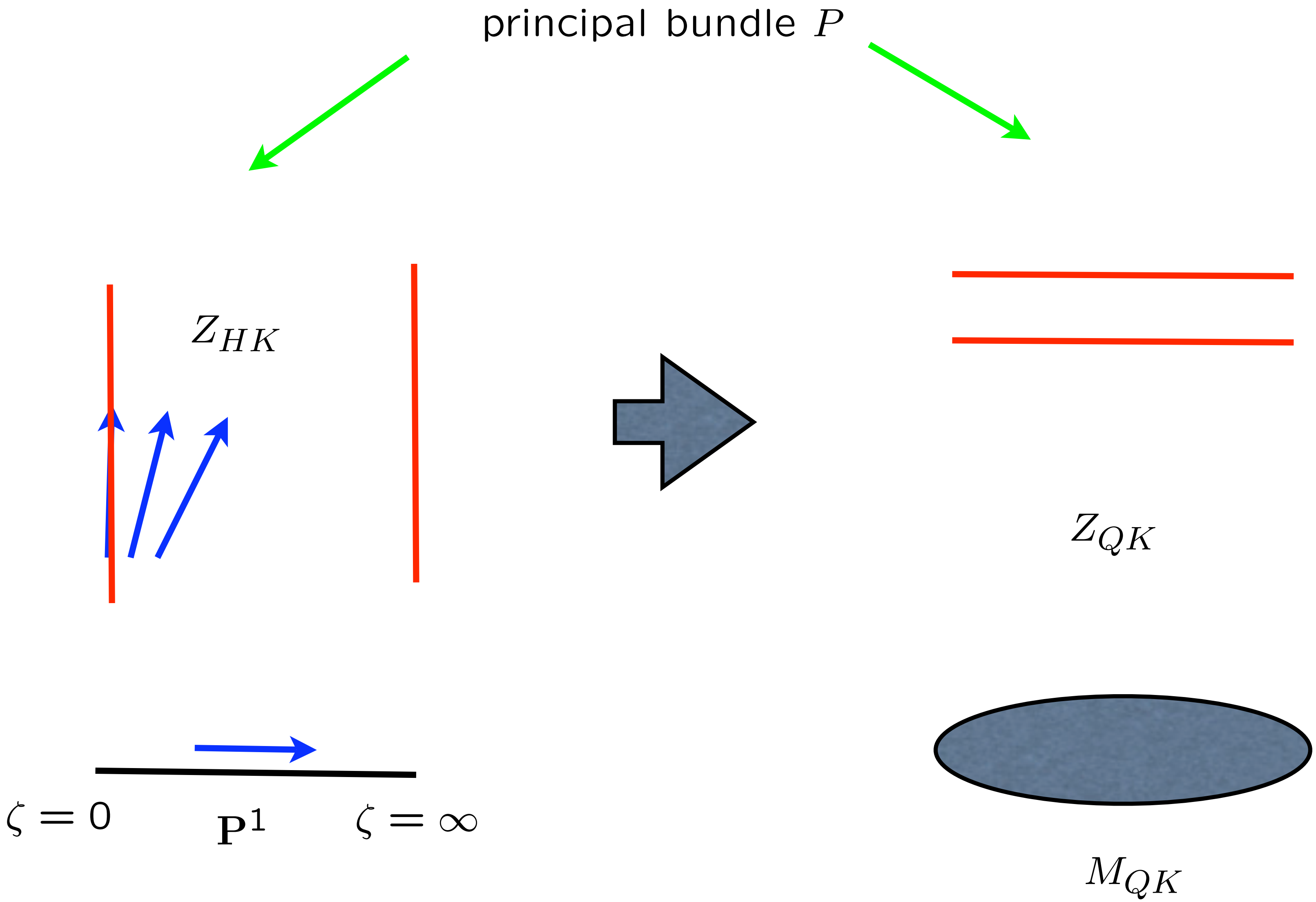
HAYDYS'S CONSTRUCTION (VIA TWISTORS)

- Z_{HK} hyperkähler twistor space, $\mathbf{C}^*$ action

- holomorphic $\mathbf{C}^*$-bundle $P \to Z_{HK}$

- lift the action, take the quotient

- $Z_{QK} = P/\mathbf{C}^*$ quaternionic Kähler twistor space


principal bundle P
Z_{HK}
Z_{QK}
M_{QK}
$\zeta = 0$
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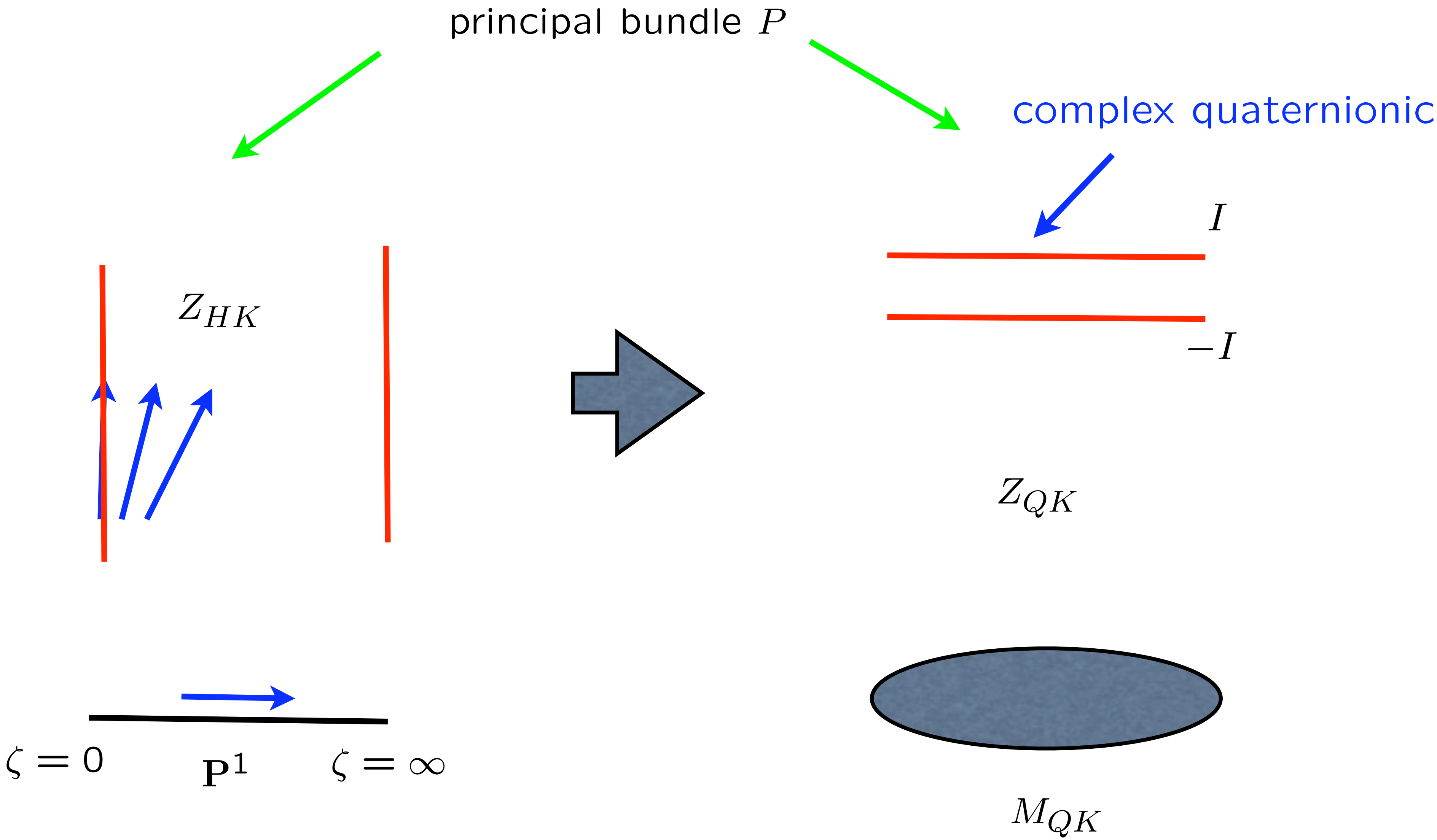

principal bundle P
complex quaternionic
I
$-I$
Z_{HK}
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M_{QK}

THE CONTACT FORM

- we want a section of $A^*(2)$ = invariant one-form on P
- $0 \to T^*(2) \to A^*(2) \to \mathcal{O}(2) \to 0$
- $H^0(Z, A^*(2)) \to H^0(Z, \mathcal{O}(2)) \to H^1(Z, T^*(2)) \to$

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- extension class for A came from

  $\theta \in H^0(D, T^*Z(2)) \to H^1(Z, T^*) \xrightarrow{\zeta} H^1(Z, T^*(2))$

- $\Rightarrow \zeta \in H^0(Z, \mathcal{O}(2))$ lifts to $A^*(2)$.

QK TO HK

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- value of moment map in $\mathbf{R}^3$ fixed by $SO(2) \subset SO(3)$ acting on Swann bundle

- $\Rightarrow$ circle action on $\hat{M}$

EXAMPLE

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- circle action on cotangent fibres
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  (defines a hyperholomorpic line bundle)

- curvature of line bundle on zero section S^2 is SO(3)-invariant

- $\Rightarrow$ multiple of ω_1

- $\Rightarrow$ by uniqueness of hyperholomorphic extension, this is the natural line bundle associated to the circle action.

EXAMPLE: EGUCHI-HANSON METRIC ON T^*S^2

- $\mathbf{C}^*$ bundle $= \{(v, \alpha) \in V(1) \oplus V^*(1) \to \mathbf{P}^1 : \alpha(v) - \zeta = 0\}$

- lifted $\mathbf{C}^*$ action $(v, \alpha) \mapsto (\lambda v, \lambda\alpha, \lambda^2\zeta)$

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- quotient $= \mathbf{P}(V \oplus V^*)$, twistor space of $\mathbf{HP}^1 = S^4$

- section of $\mathcal{O}(2)$ is $\alpha(v)$

- complex structure on $S^4 \backslash S^1 = \mathbf{CP}^1 \times$ hyperbolic plane

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  quaternionic Kähler

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  complex quaternionic

QUESTIONS

- sign of scalar curvature $\sim$ sign of moment map

- completeness for HK $\Rightarrow$? for QK

- which is more important, the quaternionic Kähler structure or the complex quaternionic structure?