Symmetries and Scattering Amplitudes

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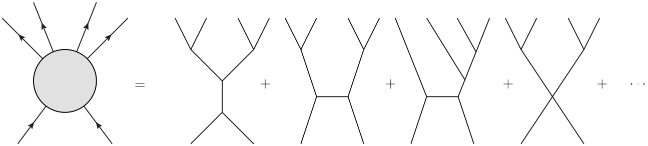
Introduction

This talk is about scattering amplitudes in four-dimensional gauge theories.



$$A(1^-2^-3^+\dots6^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle 61\rangle} \qquad \langle 12\rangle \sim \sqrt{2p_1\cdot p_2}$$

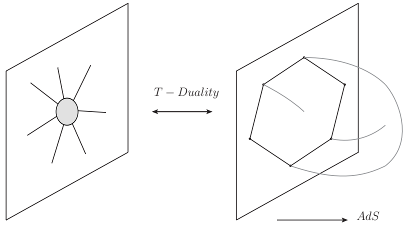
Scattering amplitudes are fundamental observables and exhibit a tremendous amount of structure.

$\mathcal{N} = 4$ Super Yang-Mills Theory

$\mathcal{N} = 4$ SYM is the most symmetric gauge theory in four dimensions.

AdS/CFT

$$\mathcal{N} = 4 \text{ SYM} \quad \Longleftrightarrow \quad \text{Type IIB strings on } \mathrm{AdS}_5 \times \mathrm{S}^5$$

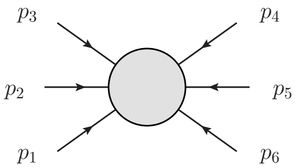


The theory is integrable in the planar limit $N \to \infty$.

- Can we compute scattering amplitudes for any value of the 't Hooft coupling λ?

Kinematics 1

Scattering amplitudes are functions of the four-momenta $\{p_i^\mu\}$ of the incoming particles.



These variables are constrained by

$$1)\quad \sum_i p_i^\mu = 0 \qquad\qquad 2)\quad p_i^2 = 0$$

encoding momentum conservation and on-shell condition.

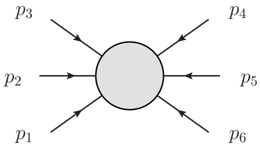
- How can we solve these constraints?

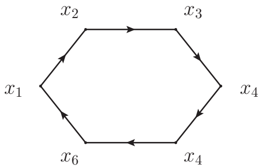
Kinematics II

- Solve momentum conservation by dual coordinates

$$p_i^\mu = x_{i+1}^\mu - x_i^\mu$$

forming a null polygon.





- Solve null condition by introducing spinors

$$p^{\alpha\dot{\alpha}} = p^\mu \sigma_\mu^{\alpha\dot{\alpha}} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}$$

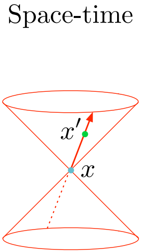
$$\det p^{\alpha\dot{\alpha}} = 0 \quad \Rightarrow \quad p^{\alpha\dot{\alpha}} = \lambda^\alpha \bar{\lambda}^{\dot{\alpha}}$$

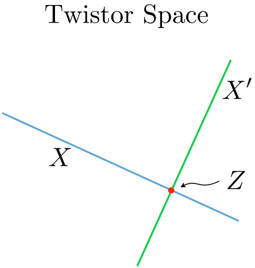
Twistor Space

Twistor space is $\mathbb{CP}^3$ with homogeneous coordinates $Z^A = (\lambda_\alpha, \mu^{\dot\alpha})$.

Incidence Relations

$$\mu^{\dot\alpha} = i x^{\alpha\dot\alpha} \lambda_\alpha$$

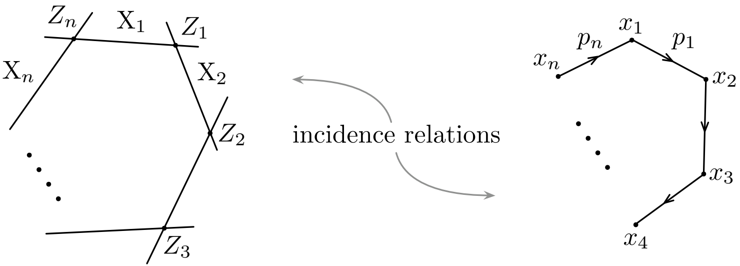




Conformal structure of space-time $\leftrightarrow$ Complex structure on twistor space

Momentum Twistors Hodges

Any ordered set $(Z_1, \ldots, Z_n)$ in twistor space determines a null polygon.



- For $\mathcal{N} = 4$ SYM promote twistor space to superspace $\mathbb{CP}^{3|4}$ with homogeneous coordinates

$$\mathcal{Z}^{\mathcal{A}} = (Z^A, \chi^a)$$

Examples

Removing momentum and supermomentum conserving delta-functions:

- The tree-level MHV amplitude is

$$A_n^{\mathrm{MHV}} = \frac{1}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

- All other amplitudes are of the form

$$A = A_n^{\mathrm{MHV}} \times \sum_{k=0} M_{n,k}$$

where $M_{n,k}$ is of order χ^{4k} in the fermions.

Symmetries of Tree-level Amplitudes

Tree amplitudes are invariant under superconformal and *dual* superconformal symmetries.

$$J^{\mathcal{A}}{}_{\mathcal{B}} = \sum_i \mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{B}}}$$

$$J^{(1)\,\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j} (-i)^{\mathcal{C}} \left(Z_i^{\mathcal{A}} \frac{\partial}{\partial Z_i^{\mathcal{C}}} Z_j^{\mathcal{C}} \frac{\partial}{\partial Z_j^{\mathcal{B}}} - (i \leftrightarrow j) \right)$$

They generate an infinite dimensional Yangian symmetry of $psu(2,2|4)$.

- This is the hallmark of integrability!

Symmetries of Loop Amplitudes

Dual superconformal symmetry is broken in loop amplitudes.

- The generators

$$D = \frac{1}{2}\left(\lambda_\alpha \frac{\partial}{\partial \lambda_\alpha} - \mu^{\dot\alpha}\frac{\partial}{\partial \mu^{\dot\alpha}}\right) \qquad K^{\alpha\dot\alpha} = \mu^{\dot\alpha}\frac{\partial}{\partial \lambda_\alpha}$$

  are broken by IR divergences.

- The generators

$$\bar{Q}_{\dot\alpha}{}^{a} = \chi^a \frac{\partial}{\partial \mu^{\dot\alpha}} \qquad S^{\alpha a} = \chi^a \frac{\partial}{\partial \lambda_\alpha}$$

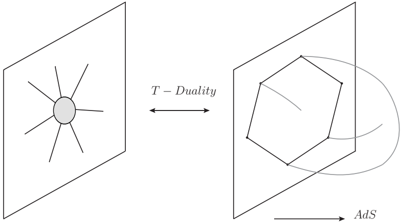
  are broken even for finite IR safe quantities.

Anomalies

Understanding the anomalies is key to unlocking the S-matrix!

Amplitude–Wilson Loop Duality I

MHV amplitudes are dual to null polygonal Wilson loops.



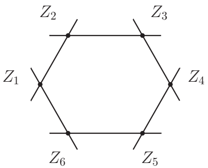
Amplitude–WL Duality

$$M_n^{\mathrm{MHV}} = \left\langle \mathrm{tr}_f \, P \exp \oint A \right\rangle$$

- Discovered at strong coupling via T-duality in AdS/CFT. Alday, Maldacena
- Subsequently observed at weak coupling. Drummond, Henn, Korchemsky, Sokatchev

Amplitude–Wilson Loop Duality II

In twistor space we have a ‘holomorphic’ Wilson loop. Mason, Skinner



Amplitude–WL Duality

$$M_n = \left\langle \, \mathrm{tr}\, P \exp \int \omega \wedge \mathcal{A} \, \right\rangle$$

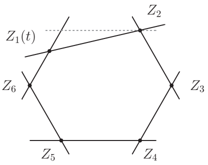
The extension beyond MHV is immediate in twistor space: $\mathbb{CP}^3 \to \mathbb{CP}^{3|4}$.

The action is then holomorphic Chern-Simons + interactions.

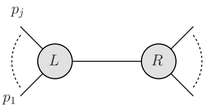
- $S(\mathcal{A}) = \int \Omega \wedge (\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}) + \lambda\,\{\text{interactions}\}$

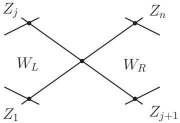
Loop Equations

Deformations of Wilson loop contour governed by loop equations.



The expectation value $\langle\, \mathrm{tr}\, P \exp \int \omega \wedge \mathcal{A} \,\rangle$ develops simple poles and factorises when the curve intersects itself.





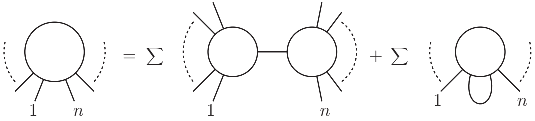
Proof of Duality Bullimore, Skinner

Wilson loops have the same singularities as scattering amplitudes!

Yangian Symmetry from the Wilson loop

Integrate over the deformation parameter to derive recursion relations.

$$M_n = M_{n-1} + \sum_{L,R} M_L \frac{1}{p^2} M_R + \frac{\int d^4 l}{l^2} M_{n+1}(l)$$



- Tree-level amplitudes are Yangian invariant Drummond, Henn, Plefka
- The *integrands* of loop corrections are Yangian invariant. Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka
- What happens to the symmetries when we perform the integrals?

The D and K Anomaly

The symmetries D and K are broken by UV divergences of cusps.

Conformal Ward identities determine that Alday, Maldacena; Drummond, Henn, Korchemsky, Sokatchev

$$M_{n,k} = e^{f(\lambda) M_n^{1-loop}} \times R_{n,k}$$

where $f(\lambda)$ is the cusp anomalous dimension.

- The ration function $R_{n,k}$ is dual conformal invariant.
- This completely fixes the answer for $n = 4, 5$.

The Ratio Function

The ratio function $R_{n,k}$ is not dual *super*-conformal invariant.

The $\bar{Q}$ Anomaly

The dual supersymmetry generators

$$\bar{Q} = \sum_i \chi_i \frac{\partial}{\partial \mu_i} \qquad \text{and} \qquad S = \sum_i \chi_i \frac{\partial}{\partial \lambda_i}$$

are broken in loop amplitudes even for the ratio function

$$\bar{Q}\, R_{n,k} \neq 0 \qquad\qquad S\, R_{n,k} \neq 0\,.$$

The anomaly has a very different origin...

The $\bar{Q}$ Anomaly Bullimore, Skinner

$\bar{Q} = \chi \frac{\partial}{\partial \mu}$ generates susy transformations of the *self-dual* theory only.

Perturbation Theory

To match perturbation theory must expand around the self-dual sector on Wilson loop side.

Self–Dual YM

$$\mathcal{L} = \mathrm{tr}\left(G^{\alpha\beta} F_{\alpha\beta} - \tfrac{\lambda}{2} G^{\alpha\beta} G_{\alpha\beta} \right) \qquad \lambda \longrightarrow 0$$

- Expectation value in self-dual theory is dual to all tree amplitudes

$$M_n^{(0)} = \int DX\ W_n\ e^{-S_1(X)}$$

- Expand around self-dual sector to generate loop expansion

$$\begin{aligned} M_n &= \int DX\ W_n\ e^{-S_1(X) - \lambda S_2(X)} \\ &= M_n^{(0)} + \lambda\, M_n^{(1)} + \lambda^2\, M_n^{(2)} + \cdots \end{aligned}$$

The Complete $\bar{Q}$ Generator

The complete $\bar{Q}$-generator has an exact one-loop correction

$$\begin{aligned} \bar{Q} &= \bar{Q}^{(0)} + \lambda\, \bar{Q}^{(1)} \\ &= \chi \frac{\partial}{\partial \mu} + ? \end{aligned}$$

- The correction does not have a geometric action on superspace.
- However its action on the component fields is well known:

$$\{\bar{Q}^{(1)}, A\} = i\,\psi$$
$$\{\bar{Q}^{(1)}, \bar{\psi}\} = -\frac{i}{2}\,[\phi\,,\,\phi]$$

- This is enough to determine the anomaly.

The $\bar{Q}$ Ward Identity

The anomaly is found by a standard Ward identity argument

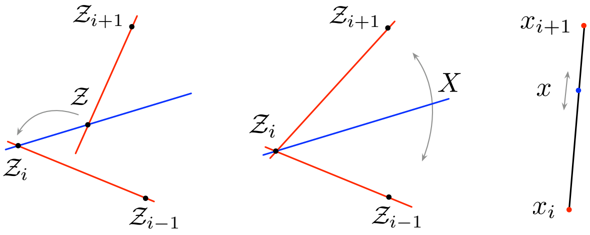
$$\begin{aligned}
\sum_{i=1}^{n} \chi_i \frac{\partial}{\partial \mu_i} M_n &= \left\langle \left\{ \bar{Q}^{(0)}, W_n \right\} \right\rangle \\
&= -\left\langle \left\{ \bar{Q}^{(1)}, W_n \right\} \right\rangle \\
&= \oint dx \, \mathrm{tr} \langle (\psi(x) + \cdots) \, \mathrm{P} \exp \oint A \rangle
\end{aligned}$$

- The field insertions are excitations of GKP string with same quantum numbers as $\bar{Q}$.
- Connection to AdS/CFT and Integrability.

Descent Equations Bullimore, Skinner

In twistor space all excitations are repackaged into a universal collinear limit operation

$$\sum_{i=1}^{n} \chi_i \frac{\partial}{\partial \mu_i} M_{k,n} = f(\lambda) \sum_i \int_{\Gamma_i} d^{2|3}Z \; M_{n+1,k+1}(Z)$$



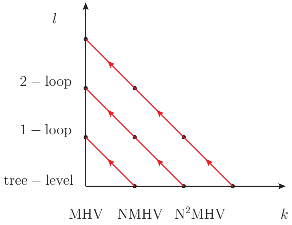
- Expand in λ and compute loop corrections from tree amplitudes!
- However, both sides are divergent and need regularization...

The Ratio Function Caron-Huot, He

Divergences are controlled by D/K anomaly $M_{n,k} = e^{-\Gamma(\lambda) M_n^{1\text{–}loop}} \times R_{n,k}$.

Derive equation for the ratio function

$$\sum_i \chi_i \frac{\partial}{\partial Z_i} R_{n,k} = f(\lambda) \sum_i \int_{\Gamma_i} d^{2|3} Z \left(R_{n+1,k+1}(Z) - R_{n,k} R^{\text{tree}}_{n+1,1}(Z) \right)$$



- Manifest transcendentality 2ℓ for ratio function at ℓ-loops.
- Powerful computational tool for small k.

Summary

- Compute scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- Scattering amplitudes are dual to supersymmetric null polygonal Wilson loops.
- Yangian symmetry at tree-level.
- Symmetries broken at loop-level but anomalies are useful.
- New progress in understanding $\bar{Q}$ and S anomalies.
- Do these anomalies completely determine the ratio functions $R_{k,n}$?
- Connection to AdS/CFT and Integrability?