HYPERKÄHLER AND QUATERNIONIC KÄHLER GEOMETRY

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UK-Japan Winter School

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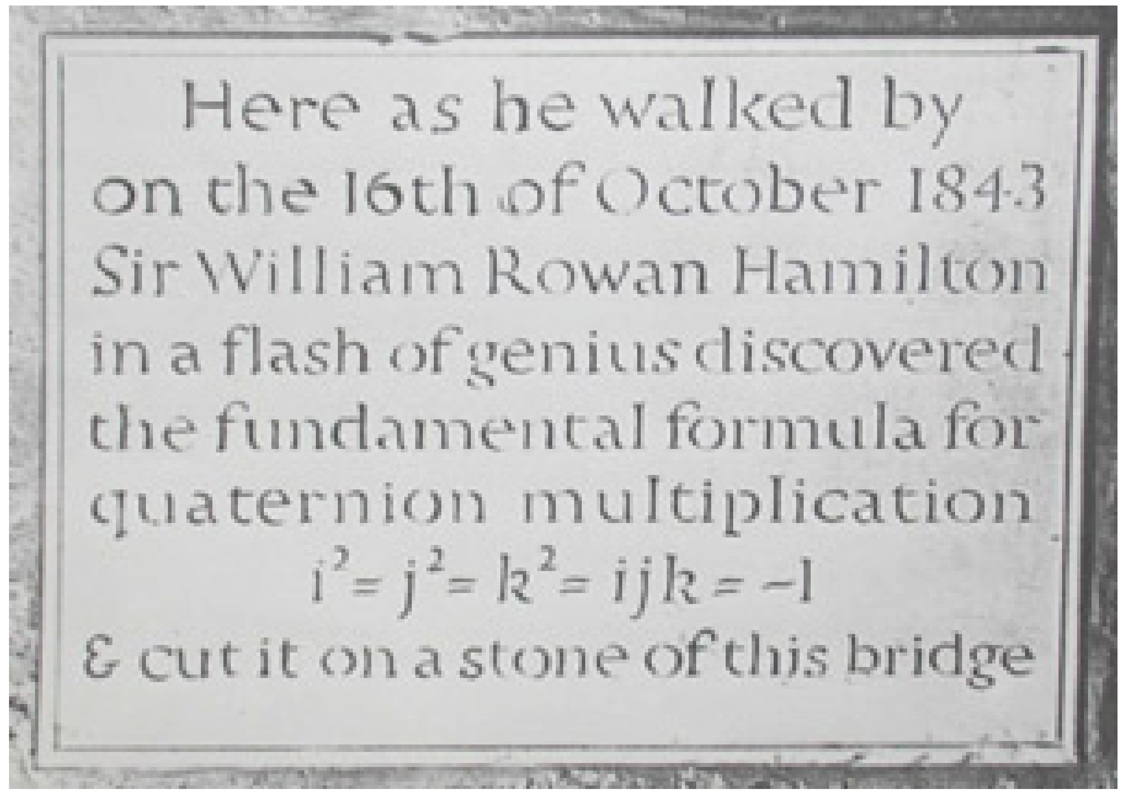
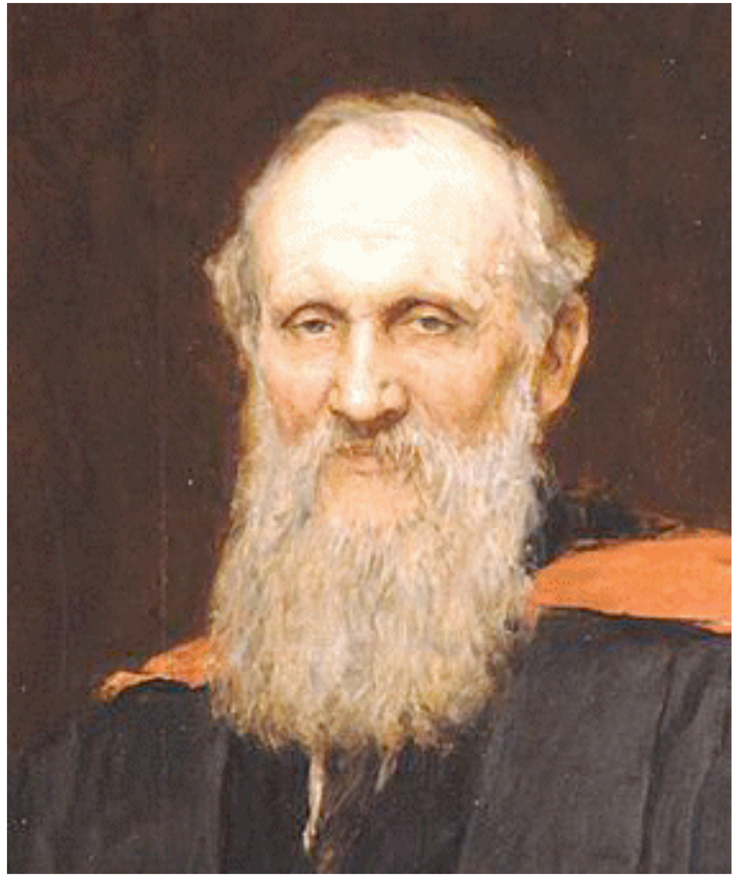
16th October 1843


Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge



"Quaternions came from Hamilton after his best work had been done, and though beautifully ingenious, they have been an unmixed evil to those who have touched them in any way"

Lord Kelvin 1890

GEOMETRY OVER THE QUATERNIONS

- $q \in \mathbf{H}$ quaternions $q = x_0 + ix_1 + jx_2 + kx_3$
- algebraic variety? $f(q_1, \ldots, q_n) = 0$
- $q^2 + 1 = 0$: 2-sphere $q = ix_1 + jx_2 + kx_3$, $x_1^2 + x_2^2 + x_3^2 = 1$

- submanifold $M \subset \mathbf{H}^n$

- $T_x M \subset \mathbf{H}^n$

- $T_x M$ quaternionic for all $x \in M \Rightarrow M = \mathbf{H}^m \subset \mathbf{H}^n$

INTRINSIC DIFFERENTIAL GEOMETRY

- quaternionic structure on the tangent bundle T
- affine connection $\nabla_X Y$
- zero torsion $\nabla_X Y - \nabla_Y X = [X, Y]$

- $\mathbf{H}^n$ n-dimensional quaternionic vector space

- left action by $GL(n, \mathbf{H})$

- commutes with right action of $\mathbf{H}$

- $GL(n, \mathbf{H}) \cdot \mathbf{H}^*$

- metric $\Leftrightarrow$ maximal compact subgroup

- $Sp(n) \cdot Sp(1) \subset GL(n, \mathbf{H}) \cdot \mathbf{H}^*$

- Levi-Civita connection ∇ : unique torsion-free connection preserving metric

- **Quaternionic Kähler** $\Leftrightarrow$ ∇ preserves quaternionic structure

- $GL(n, \mathbf{H})$ action of $\mathbf{H}$ on tangent bundle T

- $I, J, K \in \mathrm{End}(T)$ such that $I^2 = J^2 = K^2 = IJK = -1$

- metric $Sp(n) \subset GL(n, \mathbf{H})$

- Levi-Civita connection ∇ : unique torsion-free connection preserving metric

- **Hyperkähler** $\Leftrightarrow$ ∇ preserves I, J, K

- $SL(n, \mathbf{H}) \cdot U(1)$ action of $\mathbf{C}$ on tangent bundle T

- if a torsion-free connection ∇ preserves this structure, it is unique

- **complex quaternionic** – complex manifold

- D Joyce, *The hypercomplex quotient and the quaternionic quotient*, Math Ann **290** (1991) 323–340.

- $SL(n, \mathbf{H}) \cdot U(1)$

- $SL(1, \mathbf{H}) \cdot U(1) = Sp(1) \cdot U(1) = SU(2) \cdot U(1) = U(2)$

- for $n = 1$ **complex quaternionic** = Kähler complex surface with zero scalar curvature

- $n > 1$ complex quaternionic is non-metric

A.Haydys, *Hyperkähler and quaternionic Kähler manifolds with S^1 symmetries,* Jour.Geom.Phys. **58** (2008) 293–306.

S.Alexandrov, D.Persson and B.Pioline, *Wall-crossing, Rogers dilogarithm, and the QK/HK correspondence,* arXiv 1110.0466

A.Neitzke *On a hyperholomorphic line bundle over the Coulomb branch*, arXiv 1110.1619

- **Lecture 1** Quotients and moduli spaces

- **Lecture 2** Twistor theory

- **Lecture 3** The hyperkähler/quaternionic Kähler correspondence

THE HYPERKÄHLER QUOTIENT

- hyperkähler manifold M^{4k}

- complex structures I, J, K + metric g

- $\Rightarrow$ Kähler forms $\omega_1, \omega_2, \omega_3$

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- complex structures I, J, K + metric g

- $\Rightarrow$ Kähler forms $\omega_1, \omega_2, \omega_3$

- $\omega_i : T \to T^*$, $K = \omega_1^{-1}\omega_2$ etc.

- Lie group G acting on M, fixing $\omega_1, \omega_2, \omega_3$

- $a \in \mathfrak{g}$ vector field X_a

- $d(i_{X_a}\omega_i) + i_{X_a} d\omega_i = \mathcal{L}_{X_a}\omega_i = 0$

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- $d(i_{X_a}\omega_i) + \cancel{i_{X_a} d\omega_i} = \mathcal{L}_{X_a}\omega_i = 0$

- moment map $i_{X_a}\omega_i = d\mu_i^a$

- $\mu : M \to \mathfrak{g}^* \otimes \mathbf{R}^3$
- If G acts properly and freely on $\mu^{-1}(0)$ then...
- ... the quotient metric on $\mu^{-1}(0)/G$ is **hyperkähler**...
- ... of dimension $\dim M - 4 \dim G$

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- ... of dimension $\dim M - 4 \dim G$

EXAMPLE

- $M = \mathbf{H}^n = \mathbf{C}^n + j\mathbf{C}^n$ flat hyperkähler manifold

$$\omega_1 = \frac{i}{2}(dz_k \wedge d\bar{z}_k + dw_k \wedge d\bar{w}_k) \qquad \omega_2 + i\omega_3 = dz_k \wedge dw_k$$

- $G = U(1)$ action $u \cdot (z, w) = (uz, u^{-1}w)$

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- $\mu(z, w) = (z_k\bar{z}_k - w_k\bar{w}_k, z_k w_k) + const. \in \mathbf{R} \times \mathbf{C} = \mathbf{R}^3$

choice

- $\mu(z,w) = (z_k\bar{z}_k - w_k\bar{w}_k, z_k w_k) + (1,0) \in \mathbf{R} \times \mathbf{C} = \mathbf{R}^3$

- $\mu^{-1}(0) : \|z\|^2 - \|w\|^2 + 1 = 0$ and $z_k w_k = 0$

- $\mu(z,w) = (z_k\bar{z}_k - w_k\bar{w}_k, z_k w_k) + (1,0) \in \mathbf{R} \times \mathbf{C} = \mathbf{R}^3$

- $\mu^{-1}(0) : \|z\|^2 - \|w\|^2 + 1 = 0$ and $z_k w_k = 0$

- $w \neq 0 \Rightarrow$ projection $\mu^{-1}(0) \to \mathbf{C}P^{n-1}$

- $\mu^{-1}(0)/U(1) \cong T^*\mathbf{C}P^{n-1}$

Calabi metric, Eguchi-Hanson (n=2)

EXAMPLE

- $M = \mathbf{H} + \mathbf{H}$ and $G = \mathbf{R}$

- action $t \cdot (q_1, q_2) = (e^{it} q_1, q_2 + t)$

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- $\mu^{-1}(0) : |z_1|^2 - |w_1|^2 = \operatorname{im} z_2$ and $z_1 w_1 = w_2$

- $\mu^{-1}(0)/\mathbf{R} \cong \mathbf{C}^2$, coordinates (z_1, w_1)

Taub-NUT metric

NJH, A. Karlhede, U. Lindström & M. Roček, *Hyperkähler metrics and supersymmetry*, Comm. Math. Phys. **108** (1987), 535–589.

K.Galicki & H.B Lawson Jr. *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. **282** (1988) 121.

THE SWANN BUNDLE OF A QUATERNIONIC KÄHLER MANIFOLD

- $Sp(n) \cdot Sp(1) \subset GL(n, \mathbf{H}) \cdot \mathbf{H}^*$

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- **Quaternionic Kähler** $\Leftrightarrow$ ∇ preserves quaternionic structure

- principal $Sp(1)$ bundle with connection

- T is a module over a bundle of quaternions (e.g. $\mathbf{H}P^n$)

- equivalently a rank 3 bundle of 2-forms $\omega_1, \omega_2, \omega_3$

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- equivalently a rank 3 bundle of 2-forms $\omega_1, \omega_2, \omega_3$

- $\nabla \omega_1 = \theta_2 \otimes \omega_3 - \theta_3 \otimes \omega_2$

- curvature $K_{23} = d\theta_1 - \theta_2 \wedge \theta_3$ etc.

- in fact $K_{23} = c\omega_1$, c constant $\sim$ scalar curvature

- $P = SO(3)$ frame bundle
- θ_i well-defined 1-forms on P
- $\dim P \times \mathbf{R}^+ = 4n + 4$

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- θ_i well-defined 1-forms on P

- $\dim P \times \mathbf{R}^+ = 4n + 4$

- define $\varphi_i = d(t\theta_i)$ ($t = \mathbf{R}^+$ coordinate)

- three closed 2-forms $\varphi_1, \varphi_2, \varphi_3$

- $T(P \times \mathbf{R}^{+}) = H \oplus V$

- on H, $\theta_i = 0$ and $dt = 0$, $\varphi_i = tc\omega_i$

- on V, $\varphi_1 = dt \wedge \theta_1 + t^2 \theta_2 \wedge \theta_3$ etc.

- algebraic relations for hyperkähler if $c > 0$

  Lorentzian version $Sp(1, n)$ if $c < 0$

EXAMPLE

- $M = \mathbf{H}P^n$ quaternionic projective space

- $P = S^{4n+3} \subset \mathbf{H}^{n+1}$

- $P \times \mathbf{R}^+ = \mathbf{H}^{n+1} \backslash \{0\}$

- $P \times \mathbf{R}^{+}$ = **Swann bundle** or **hyperkähler cone**

- G preserves quaternionic Kähler structure $\Rightarrow$ induced action on P preserves $\varphi_1, \varphi_2, \varphi_3$

- Quaternionic Kähler quotient $\Leftrightarrow$ hyperkähler quotient on Swann bundle

EXAMPLE

- $M = Sp(2,1)/Sp(2) \times Sp(1)$ and $G = \mathbf{R}$
- $\mathbf{R} = SO(1,1) \subset Sp(1,1) \subset Sp(2,1)$
- Quotient = deformation of hyperbolic metric on B^4
- self-dual Einstein

HYPERKÄHLER MODULI SPACES

- Higgs bundles

- magnetic monopoles

- Higgs bundles

A.Kapustin & E.Witten, *Electric-magnetic duality and the geometric Langlands program* Commun. Number Theory Phys. **1** (2007) 1-236

D.Gaiotto, G.W. Moore & A.Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory* Comm. Math. Phys. **299** (2010)163-224.

- magnetic monopoles

M.Atiyah & NJH, *The geometry and dynamics of magnetic monopole* Princeton University Press, Princeton, NJ, 1988.

R.Bielawski, *Monopoles and clusters*, Comm. Math. Phys. **284** (2008),675-712.

HIGGS BUNDLES

- Σ compact Riemann surface

- V smooth vector bundle with Hermitian metric

- $\mathcal{A} =$ infinite-dimensional affine space of $\bar{\partial}$-operators on V

$$\bar{\partial}_A : \Omega^0(V) \to \Omega^{0,1}(V) \qquad \bar{\partial}_A(fs) = f\bar{\partial}_A(s) + \bar{\partial} f s$$

- $\bar{\partial}_A - \bar{\partial}_B \in \Omega^{0,1}(\mathrm{End}(V))$

- $M = \mathcal{A} \times \Omega^{1,0}(\mathsf{End}(V))$

- $T_A M = \Omega^{0,1}(\mathsf{End}(V)) \oplus \Omega^{1,0}(\mathsf{End}(V))$

- Hermitian form $\omega_1 \sim \int_\Sigma (\mathsf{tr}\, aa^* + \mathsf{tr}\, \phi\phi^*)$

$$\omega_2 + i\omega_3 \sim \int_\Sigma \mathsf{tr}\, a\phi$$

- flat hyperkähler manifold

- G = group of $U(n)$ gauge transformations
- $\mathfrak{g} = \{\psi \in \Omega^0(\mathrm{End}(V)), \psi^* = -\psi\}$
- $\mathfrak{g}^* = \{\omega \in \Omega^2(\mathrm{End}(V)), \omega^* = -\omega\}$

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- moment map $\mu(\bar{\partial}_A, \Phi) = (F_A + [\Phi, \Phi^*], \bar{\partial}_A \Phi)$

- F_A= curvature of Hermitian connection with $\nabla^{0,1} = \bar{\partial}_A$

- $\mu(\bar{\partial}_A, \Phi) = (F_A + [\Phi, \Phi^*], \bar{\partial}_A \Phi)$

- hyperkähler quotient = $\mu^{-1}(0)/G$ = moduli space

- $\bar{\partial}_A \Phi = 0$ = holomorphic Higgs field $\Phi \in \mathrm{End}(V) \otimes K$

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- $\nabla + \Phi + \Phi^*$ connection

$$F = [\nabla^{1,0} + \Phi, \nabla^{0,1} + \Phi^*] = 0$$

flat $GL(n, \mathbf{C})$ connection

- $\mu(\bar{\partial}_A, \Phi) = (F_A + [\Phi, \Phi^*], \bar{\partial}_A \Phi)$

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complex structure I

- $\nabla + \Phi + \Phi^*$ connection

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flat $GL(n, \mathbf{C})$ connection

complex structure J

MAGNETIC MONOPOLES

- principal $SU(2)$ bundle on $\mathbf{R}^3$
- $\mathcal{A} =$ infinite-dimensional affine space of connections
- $\phi =$ Higgs field $\in \Omega^0(\mathfrak{su}(2))$
- $T_A(\mathcal{A}, \phi) = \Omega^1(\mathfrak{su}(2)) \oplus \Omega^0(\mathfrak{su}(2))$
- $I(a_1 dx_1 + a_2 dx_2 + a_3 dx_3 + \psi) = \psi dx_1 - a_3 dx_2 + a_2 dx_3 - a_1$

- $\|\phi\| \to 1$ as $r \to \infty$

- $\Rightarrow$ flat hyperkähler metric

- $G = SU(2)$ gauge transformations

- moment map $\mu(A, \phi) = F_A - *\nabla\phi$

- $\|\phi\| \to 1$ as $r \to \infty$

- $\Rightarrow$ flat hyperkähler metric

- $G = SU(2)$ gauge transformations

- moment map $\mu(A, \phi) = F_A - *\nabla\phi$

- $\mu^{-1}(0)/G$ = moduli space of solutions to

  Bogomolny equations $F_A = *\nabla\phi$

THE KÄHLER POTENTIAL

- Kähler form $\omega \in \Omega^{1,1}$
- $d\omega = 0$
- locally $\omega = dd^c f = dIdf$
- f Kähler potential

- hyperkähler $\omega_1, \omega_2, \omega_3$

- vector field X

- $\mathcal{L}_X\omega_1 = 0, \quad \mathcal{L}_X\omega_2 = \omega_3, \quad \mathcal{L}_X\omega_3 = -\omega_2$

- non-trivial representation of $U(1)$ on $\mathbf{R}^3$

EXAMPLES

- Higgs bundles: $(A, \Phi) \mapsto (A, e^{i\theta}\Phi)$

- magnetic monopoles: rotation about x_1-axis

- $\mathcal{L}_X \omega_1 = 0$ moment map μ

- $i_X \omega_1 = d\mu$

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- $K d\mu(Y) = -\omega_1(X, KY) = -g(IX, KY) =$

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- $K d\mu(Y) = -\omega_1(X, KY) = -g(IX, KY) =$

$$= g(KIX, Y) = g(JX, Y) = i_X \omega_2(Y)$$

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- $dKd\mu = d(i_X \omega_2) = \mathcal{L}_X \omega_2 = \omega_3$

Kähler potential

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Kähler potential