Families of $K3$ surfaces in the smooth Fano 3-folds

Makiko Mase
Tokyo Metropolitan University

1 Introduction

Variety refers a complete, complex, connected, irreducible and reduced variety.

1.1 Motivations

The reasons why we are interested in families of $K3$ surfaces are : firstly, the "$K3$ fibration", which form a family of $K3$ surfaces, is a special case of Iitaka fibration ; secondly, mirror symmetry, which recently place families of $K3$ surfaces as one of the central objects to study, leads us to compute Picard-Fuchs differential equation and some invariants related to the number of rational curves on a $K3$ surface, etc.

Remark 1 (1) If $\mathcal{F}$ is a family of $K3$ surfaces, $\mathcal{F}$ contains not only Gorenstein $K3$ surfaces (see definition below) but also degenerate $K3$ surfaces. Once we get a degenerate $K3$ surface, we can consider mirror near this point.
(2) In spite of a fact that the mirror symmetry theory of Calabi-Yau manifolds in toric varieties is well-known, the mirror of Calabi-Yau manifolds in non-toric varieties is not enough studied. So we would like to consider mirror for $K3$ surfaces in non-toric varieties possibly via mirror for $K3$s in toric varieties.

Thirdly, we would like to extend the following picture (which is a well-known fact from algebraic/projective geometry) to $K3$ surfaces in *non-toric* 3-folds:

Motivating Example. Generic members in families of plane cubics and $(2,2)$-curves in $\boldsymbol{P}^1 \times \boldsymbol{P}^1$are birationally corresponding: the map f in the Figure 1 is a blow-up of $\boldsymbol{P}^2$ at two points P, Q and g is a blow-up at a point R. Any plane cubic C passing through the points P, Q are sent to a $(2,2)$-curve C' in $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ passing through the point R by the birational map $g \circ f^{-1}$ and vice versa, where $\widetilde{C}$ in Figure 1 denotes the strict transform of C or C' by f or g. The complete anticanonical linear systems of $\boldsymbol{P}^2$ and $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ have a common sublinear system which is the complete anticanonical linear system of the del Pezzo surface of degree seven, correspondingly, the monomial polytopes for $\boldsymbol{P}^2$ and $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ have the common polytope.

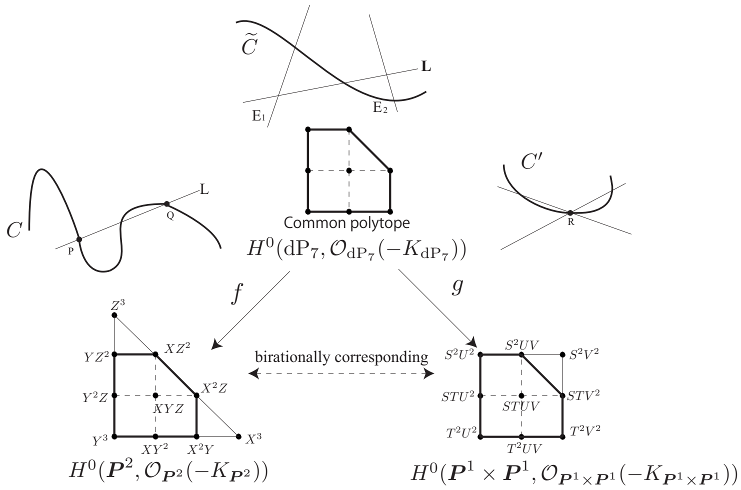

Figure 1: Motivating birational correspondence

1.2 Setting

Definition 1 (1) A variety S of two-dimensional is called a *Gorenstein $K3$ surface* if $h^1(\mathcal{O}_S) = 0, K_S \sim 0$ and S has at worst ADE singularities. When a Gorenstein $K3$ surface is nonsingular, we refer it simply a $K3$ *surface*. (Due to the existence of crepant resolution for ADE singularities, a Gorenstein $K3$ surface is birational to a $K3$ surface.)
(2) For a $K3$ surface S, the *Picard lattice* $\mathrm{Pic}(S)$ of S is the Picard group of S with a cup product. (Since the irregularity is zero, the Picard group is naturally embedded into the $K3$ lattice.)

Definition 2 A 3-dimensional algebraic variety X is called a *smooth Fano 3-fold* if X is nonsingular and the anticanonical divisor $-K_X$ is ample.

Smooth Fano 3-folds are classified into 88 classes in case the second Betti number ≥ 2 by Mori-Mukai [7][8], and 18 classes in case of toric by Batyrev [1] and Watanabe-Watanabe [9].

Let X be a smooth Fano 3-fold. It is proved by Šokurov that any general anticanonical member S in the complete anticanonical linear system $|-K_X|$ of X is smooth. It is easily shown that a general member $S \in |-K_X|$ is a $K3$ surface by using adjunction formula, Lefschetz's hyperplane section theorem and Kodaira's vanishing theorem. Thus the complete anticanonical linear system $|-K_X|$ parametrises a family of $K3$ surfaces in the smooth Fano 3-fold X.

Definition 3 Let $\mathcal{F}$ be a family of (Gorenstein) $K3$ surfaces in a smooth Fano 3-fold. The *Picard lattice* $\mathrm{Pic}(\mathcal{F})$ of $\mathcal{F}$ is the Picard lattice of generic member in $\mathcal{F}$.

1.3 Problem

Let us consider the following problem:

Problem. Let $\mathcal{F}$ and $\mathcal{G}$ be families of $K3$ surfaces in the different classes of smooth Fano 3-folds. If $\mathrm{Pic}(\mathcal{F}) \simeq \mathrm{Pic}(\mathcal{G})$, then are the families $\mathcal{F}$ and $\mathcal{G}$ generically birationally corresponding, that is, for any generic member in $\mathcal{F}$ does there exist a generic member in $\mathcal{G}$ that are birational ?

In particular, we are interested in considering the following specific families: let l, C be a line and a smooth plane cubic in $\boldsymbol{P}^3$. We may assume without loss of generality that l and C are lying on a plane H in $\boldsymbol{P}^3$. Let

$$\sigma : X' \to \boldsymbol{P}^3, \quad \pi : X \to \boldsymbol{P}^3$$

be the blow-ups of $\boldsymbol{P}^3$ along l, C with the exceptional divisors D, E, respectively. Let $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) be the family of Gorenstein $K3$ surfaces in X' (resp. X). Denote by $M_i := \mathrm{Pic}(\mathcal{F}_i)$ the Picard lattice of the family $\mathcal{F}_i$ and Ω_i be the moduli space of ample M_i-polarised $K3$ surfaces that are the minimal models of Gorenstein $K3$ surfaces in $\mathcal{F}_i$, $i = 1, 2$. For the moduli space of lattice-polarised $K3$ surfaces, we refer Dolgachev [4].

Remark 2 (1) The smooth Fano 3-fold X' is toric and X is non-toric.
(2) The Picard lattices $\mathrm{Pic}(\mathcal{F}_1)$ and $\mathrm{Pic}(\mathcal{F}_2)$ are isometric to a lattice $M := \left(\boldsymbol{Z}^2, \begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}\right)$ of rank 2 and the index $(1,1)$. Indeed, since $\mathrm{Pic}(\mathcal{F}_1) = \boldsymbol{Z}\sigma^*H \oplus \boldsymbol{Z}(\sigma^*H - D)$ and $\mathrm{Pic}(\mathcal{F}_2) = \boldsymbol{Z}\pi^*H \oplus \boldsymbol{Z}E$, the intersection matrices are given by

$$\begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}.$$

Thus we must ask the following question as a special case of out problem:

Problem (Special case). Does there exist a birational correspondence between Gorenstein $K3$ surfaces in families $\mathcal{F}_1$ and $\mathcal{F}_2$?

Remark 3 If the Picard numbers are large enough, the statement of the problem may be proved by Nikulin's lattice theory and Torelli-type theorem for $K3$ surfaces. However, if the Picard numbers, which is the rank of the Picard lattice, are small, for example, families of $K3$ surfaces in smooth Fano 3-folds, whose Picard numbers are almost ≤ 5 it may not be true (since we are not sure that a primitive embedding of the Picard lattice into the $K3$ lattice is uniquely determined).

Remark 4 Recall Torelli-type theorem for $K3$ surfaces:
Torelli-type theorem *Two $K3$ surfaces S, S' are isomorphic as complex varieties if and only if there exists an effective Hodge isometry between $H^2(S', \boldsymbol{Z})$ and $H^2(S, \boldsymbol{Z})$.*
Here, an isometry is an isomorphism between lattices that preserves cup product and 'effective Hodge' means that the isometry preserves effective divisors (that is, curves) on S and S', and Hodge decompositions of $H^2(S', \boldsymbol{C})$ and $H^2(S, \boldsymbol{C})$.

2 Main Results

We first study the families of Gorenstein $K3$ surfaces in smooth *toric* Fano 3-folds and obtain the following result.

Theorem 1 *No families of $K3$ surfaces in smooth* toric *Fano 3-folds are birationally corresponding.*

In order to prove this result, we first compute the Picard lattices of families of Gorenstein $K3$ surfaces in smooth toric Fano 3-folds and show that these Picard lattices are mutually distinct. Hence, by the Torelli-type theorem, Theorem 1 follows.

Next we study the families $\mathcal{F}_1$ and $\mathcal{F}_2$ carefully and obtain the result as follows.

Theorem 2 *The moduli spaces Ω_1, Ω_2 are isomorphic.*

Remark 5 Theorem 2 answers Problem (Speceial case) positively. Indeed, let $\omega_1 \in \Omega_1$, then by the surjectivity of a period map, there exists a Gorenstein $K3$ surface $S' \in \mathcal{F}_1$ whose period point is ω_1. By Theorem 2, there exists a unique point $\omega_2 \in \Omega_2$, such that $\omega_1 = \omega_2$. Again by the surjectivity of a period map, there exists a Gorenstein $K3$ surface $S \in \mathcal{F}_2$ whose period point is ω_2. Hence, S' and S are birataional to each other. Therefore, there exists a correspondence between Gorenstein $K3$ surfaces in $\mathcal{F}_1$ and $\mathcal{F}_2$.

More precisely, there exists a birational surjective correspondence $(V, \mathcal{F}_1, \mathcal{F}_2)$, where

$$V := \left\{ (S, H) \;\middle|\; \begin{array}{l} H \subset \boldsymbol{P}^3 \text{ is a plane, } S \text{ is a Gorenstein } K3 \text{ surface,} \\ S \cap H \text{ is a union of a line and a smooth cubic} \end{array} \right\}.$$

Let B_N be a subset in $\boldsymbol{P}^N$, where N is 21 and 28 according to $i = 1, 2$.

$$\begin{tikzcd}
 & V \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
\mathcal{F}_1 \arrow[rr, "R"] \arrow[d, "f_1"'] & & \mathcal{F}_2 \arrow[d, "f_2"] \\
B_{28} \arrow[d, "p_1"'] & & B_{21} \arrow[d, "p_2"] \\
\Omega_1 \arrow[rr, "\varphi"] & & \Omega_2
\end{tikzcd}$$

Figure 2: Commutative diagram for Theorem 2

Sketch of the proof of Theorem 2

Step 1. Any member in $|-K_{X'}|$ (resp. $|-K_X|$) is birational to a member in $|-K_{\boldsymbol{P}^3} - l|$ (resp. $|-K_{\boldsymbol{P}^3} - C|$).

Step 2. Show that the subspace

$$\mathbf{S} := \left\{ S \in |-K_{\boldsymbol{P}^3} - C| \;\middle|\; \begin{array}{l} S \text{ is a smooth } K3 \text{ surface, there exists an} \\ \text{irreducible smooth plane cubic } C' \in |C| \\ \text{such that } C' \text{ is not isomorphic to } C \text{ and} \\ S \text{ contains } C' \text{ as a fibre of the fibration } \Phi_{|C|} \end{array} \right\}$$

is (non-empty) Zariski dense open.

Step 3. (crucial part) Show that for non-isomorphic irreducible smooth plane cubics, C and C', Gorenstein $K3$ surfaces in the families $|-K_{\boldsymbol{P}^3} - C|, |-K_{\boldsymbol{P}^3} - C'|$ are birationally corresponding.

Step 4. Remarking that the hyperplane section $S \cap H$, which is a curve of degree 4, where S is a Gorenstein $K3$ surface in $|-K_{\boldsymbol{P}^3} - l|$ or $|-K_{\boldsymbol{P}^3} - C|$, is a union of a line and an irreducible smooth plane cubic, show that Gorenstein $K3$ surfaces in $|-K_{\boldsymbol{P}^3} - l|$ and $|-K_{\boldsymbol{P}^3} - C|$ are birationally correseponding using some projective transformations of $\boldsymbol{P}^3$ and "limit technic".

Step 5. Since the period points in the moduli spaces are birational-invariant, the period points of birational Gorenstein $K3$ surfaces $S_1 \in \mathcal{F}_1$ and $S_2 \in \mathcal{F}_2$ coincide. Thus, we can constract a natural isomorphism between Ω_1 and Ω_2. $\square$

3 Application 1

We recall some basics from toric geometry.

Let $N \simeq \boldsymbol{Z}^n$ be a lattice and $M := \mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$ be the dual with a natural cup product $\langle , \rangle : M \times N \to \boldsymbol{Z}$. Denote $N_{\boldsymbol{R}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ and $M_{\boldsymbol{R}} := M \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ with the extended cup product $\langle , \rangle_{\boldsymbol{R}}$. Let Σ in $N_{\boldsymbol{R}}$ be a n-dimensional projective fan, that is, the toric variety $\boldsymbol{P}(\Sigma)$ obtained by Σ is a projective (normal) variety of dimension n. Assume that Σ has one-dimensional cones $\sigma_1, \ldots, \sigma_d$. Then there exist primitive lattice vectors $v_1, \ldots, v_d$ such that $\sigma_i = \boldsymbol{R}_{\geq 0} v_i$, $i = 1, \ldots, d$. We can associate to Σ a polytope Δ as the convex hull of $v_1, \ldots, v_d$. Denote the associated toric vatiety by $\boldsymbol{P}(\Sigma) = \boldsymbol{P}(\Delta)$ and torus-invariant divisor $\overline{\mathrm{orb}(\sigma_i)}$ on $\boldsymbol{P}(\Sigma)$ by D_i.

Definition 4 The *polar dual* Δ^* of Δ is a polytope

$$\Delta^* := \{ m \in M_{\boldsymbol{R}} \mid \langle m, u \rangle \geq -1 \ \text{ for all } \ u \in \Delta \} .$$

Remark 6 (1) The anticanonical divisor of $\boldsymbol{P}(\Delta)$ is given by

$$-K_{\boldsymbol{P}(\Delta)} = \sum_{i=1}^{d} D_i.$$

(2) For a torus-invariant Cartier divisor $D = \sum_{i=1}^{d} a_i D_i$ on $\boldsymbol{P}(\Delta)$, let

$$P_D := \{ m \in M_{\boldsymbol{R}} \mid \langle m, v_i \rangle \geq -a_i, i = 1, \ldots, d \} .$$

Then the group of global sections of the line bundle $\mathcal{O}(D)$ on $\boldsymbol{P}(\Delta)$ is given as

$$H^0(\boldsymbol{P}(\Delta), \mathcal{O}(D)) = \bigoplus_{m \in P_D \cap M} \boldsymbol{C} \chi^m,$$

where χ^m is a charachter. In particular, we have

$$H^0(\boldsymbol{P}(\Delta), \mathcal{O}(-K_{\boldsymbol{P}(\Delta)})) = \bigoplus_{m \in P_{-K_{\boldsymbol{P}(\Delta)}} \cap M} \boldsymbol{C} \chi^m,$$

where by definition,

$$P_{-K_{\boldsymbol{P}(\Delta)}} = \{ m \in M_{\boldsymbol{R}} \mid \langle m, v_i \rangle \geq -1, i = 1, \ldots, d \} = \Delta^*.$$

Hence, it is equivalent to consider the members in $H^0(\boldsymbol{P}(\Delta), \mathcal{O}(-K_{\boldsymbol{P}(\Delta)}))$ and lattice points on Δ^* via the correspondence

$$\begin{array}{ccc} H^0(\boldsymbol{P}(\Delta), \mathcal{O}(-K_{\boldsymbol{P}(\Delta)})) & \longleftrightarrow & \Delta^* \\ x_1^{m_1} x_2^{m_2} x_3^{m_3} & \longleftrightarrow & m = (m_1, m_2, m_3) \\ \text{monomials} & & \text{lattice points.} \end{array}$$

This is why Δ^* is sometimes called a monomial polytope.

Definition 5 [2] Let Δ be an integral polytope in $\boldsymbol{R}^n$ whose relative interior contains only one lattice point. Δ is called *reflexive* if the polar dual Δ^* is also integral.

The relation between reflexivity and families of $K3$ surfaces is as follows:

Theorem 3 [2] *An integral n-dimensional polytope Δ is reflexive if and only if the minimal model of irreducible anticanonical divisor of $\boldsymbol{P}(\Delta)$ is a Calabi-Yau $(n-1)$-fold.*

In the following, we only deal with the case $n = 3$.

Definition 6 [3] Let Y be a normal Gorenstein toric Fano 3-fold. Y is called a *small toric degeneration* of smooth Fano 3-folds if there exists a projective flat morphism $\pi : \mathfrak{X} \to \Delta_1 := \{t \in \boldsymbol{C} \,|\, |t| < 1\}$ with $\mathfrak{X}$ an irreducible complex manifold, such that
(1) For all $t \in \Delta_1 \backslash \{0\}$, the fibre $X_t = \pi^{-1}(t)$ is a smooth Fano 3-fold,
(2) The central fibre $X_0 = \pi^{-1}(0)$ has at worst Gorenstein terminal singularities and $X_0 \simeq Y$,
(3) For all $t \in \Delta_1$, the restriction map $\mathrm{Pic}(\mathfrak{X}) \to \mathrm{Pic}(X_t)$ of the Picard groups is an isomorphism.

Remark 7 Gorenstein terminal singularities in 3-dimensional toric varieties are known to be nodes $(XZ - YW = 0)$.

Let $\Sigma_{X'}$ in $\boldsymbol{R}^3$ (see Fig. 3) be a fan with 1-simplices generated by (see [9])

$$(1,0,0), (0,1,0), (0,0,1), (-1,-1,0), (-1,-1,-1).$$

Then $X' = \boldsymbol{P}(\Delta_{X'})$ and the polar dual $\Delta^*_{X'}$ of the polytope $\Delta_{X'} = \Delta(\Sigma_{X'})$ is a polytope with vertices

$$\begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

On the other hand, let Σ_Y in $\boldsymbol{R}^3$ (see Fig. 4) be a fan with 1-simplices generated by

$$(1,0,0), (0,1,0), (0,0,1), (-1,-1,0), (-1,0,-1), (0,-1,-1), (-1,-1,-1).$$

Then $Y := \boldsymbol{P}(\Delta_Y)$ is the small toric degeneration of the non-toric smooth Fano 3-fold X [6]. The toric Fano 3-fold Y has three nodes, and the Picard number

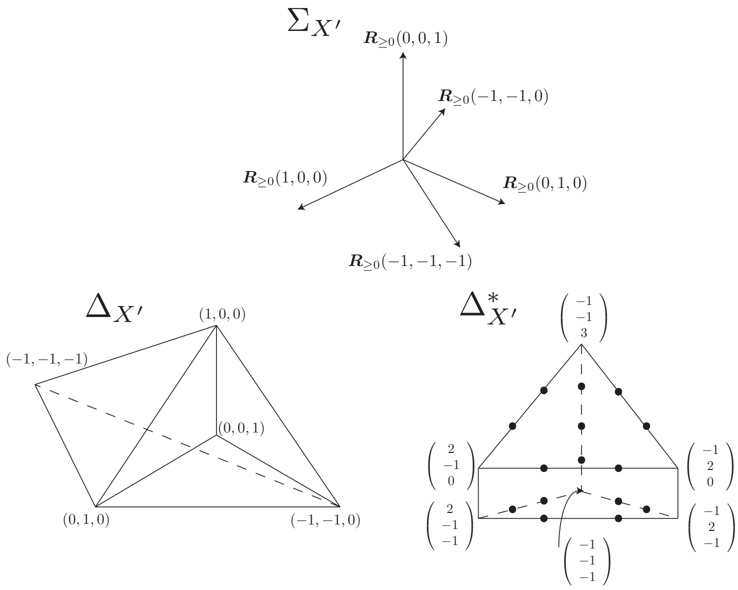

Figure 3: The fan $\Sigma_{X'}$, and convex polytopes $\Delta_{X'}$ and $\Delta^*_{X'}$.

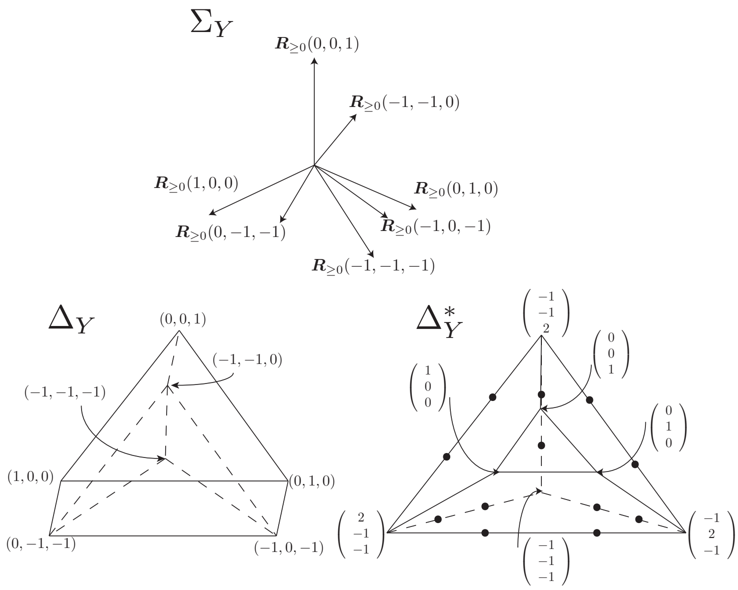

Figure 4: The fan Σ_Y, and convex polytopes Δ_Y and Δ^*_Y.

and the degree of Y are the same as those of X's. The polar dual Δ_Y^* of the polytope $\Delta_Y = \Delta(\Sigma_Y)$ is a polytope with vertices

$$\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The fan $\Sigma_{\boldsymbol{P}^3}$ defining the projective space $\boldsymbol{P}^3$ has as is well-known four 1-simplices

$$\boldsymbol{R}_{\geq 0}(1,0,0),\ \boldsymbol{R}_{\geq 0}(0,1,0),\ \boldsymbol{R}_{\geq 0}(0,0,1),\ \boldsymbol{R}_{\geq 0}(-1,-1,-1).$$

The fan $\Sigma_{X'}$ is obtained by adding a 1-simplex

$$\boldsymbol{R}_{\geq 0}(-1,-1,0) = \boldsymbol{R}_{\geq 0}(0,0,1) + \boldsymbol{R}_{\geq 0}(-1,-1,-1)$$

to the fan $\Sigma_{\boldsymbol{P}^3}$. Moreover, the fan Σ_Y is obtained by adding three 1-simplices

$$\begin{array}{rcl} \boldsymbol{R}_{\geq 0}(-1,-1,0) & = & \boldsymbol{R}_{\geq 0}(0,0,1) + \boldsymbol{R}_{\geq 0}(-1,-1,-1), \\ \boldsymbol{R}_{\geq 0}(-1,0,-1) & = & \boldsymbol{R}_{\geq 0}(0,1,0) + \boldsymbol{R}_{\geq 0}(-1,-1,-1), \\ \boldsymbol{R}_{\geq 0}(0,-1,-1) & = & \boldsymbol{R}_{\geq 0}(1,0,0) + \boldsymbol{R}_{\geq 0}(-1,-1,-1) \end{array}$$

to the fan $\Sigma_{\boldsymbol{P}^3}$. This means (see Figure 5) that X' is obtained by σ the blow-up a line (as is defined), say $l = l_1$ that is passing two points $P = (0:0:1:0)$ and $Q = (0:0:0:1)$, and Y is obtained by τ the blow-up along three lines l_1, l_2, l_3, where l_2 is passing through Q and $R = (0:1:0:0)$ and l_3 through R and P.

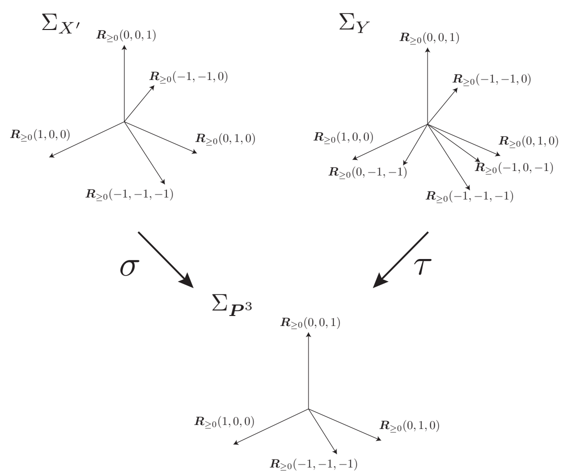

Figure 5: Δ_Y^* is a subpolytope of $\Delta_{X'}^*$.

It is easily observed that the polytope Δ_Y^* is a subpolytope of $\Delta_{X'}^*$ (see Fig. 6). Hence, there exists a unique monomial transformation

$$\begin{array}{cccccc} H^0(X', \mathcal{O}_{X'}(-K_{X'})) & W^4 & WX^3 & WZ^3 & WY^3 & XY^2Z \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ H^0(Y, \mathcal{O}_Y(-K_Y)) & W^4 & WX^3 & WZ^3 & WY^3 & XY^2Z. \end{array}$$

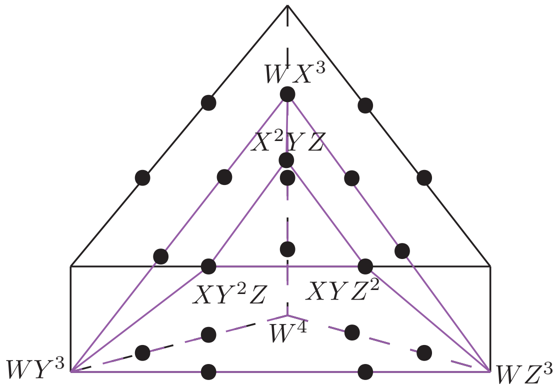

Figure 6: Δ_Y^* is a subpolytope of $\Delta_{X'}^*$.

up to permutation of three variables X, Y, and Z.

The polytope Δ_Y is reflexive and the minimal models of irreducible anti-canonical members of Y are $K3$ surfaces. Let $\mathcal{F}_3$ be the family of Gorenstein $K3$ surfaces parametrised by the complete anticanonical linear system $|-K_Y|$ of Y.

Conjecture The family $\mathcal{F}_3$ of Gorenstein $K3$ surfaces is generically birationally corresponding to the families $\mathcal{F}_1, \mathcal{F}_2$.

As is seen from the figure of polytopes, the monomial $WXYZ$ is contained in Δ_Y^* and $\Delta_{X'}^*$. Note that the generic members in $|-K_Y|$ are Gorenstein $K3$ surfaces since they do not pass any nodes on Y.

4 Application 2

There exists another family of $K3$ surfaces in a smooth Fano 3-fold whose Picard lattice is isometric to the lattice M.

Let $K := (3) \cap (3)$ be a smooth irreducible curve in $\boldsymbol{P}^3$ which is a general intersection of two cubic hypersurfaces in $\boldsymbol{P}^3$. Let $\tau : X'' \to \boldsymbol{P}^3$ be the blow-up of $\boldsymbol{P}^3$ along the curve K with the exceptional divisor F. Then it is known that X'' is a smooth (non-toric) Fano 3-fold [7]. For a generic member $S \in |-K_{X''}|$, since $S \sim -K_{X''} = \tau^*(-K_{\boldsymbol{P}^3} - F) = 4\tau^*H - F$, we have

$$\tau^*H^2|_{-K_{X''}} = 4H^3 = 4, \qquad \tau^*H.F|_{-K_{X''}} = -\tau^*H.F = \deg(K) = 3 \cdot 3 = 9,$$

$$F^2|_{-K_{X''}} = 2g(K) - 2 = 2\left\{\tfrac{1}{2} \cdot 3 \cdot 3(3+3-4) + 1\right\} - 2 = 18.$$

Hence, the Picard lattice $\mathrm{Pic}(S)$ has an intersection matrix (see also [5])

$$\begin{pmatrix} 4 & 9 \\ 9 & 18 \end{pmatrix},$$

which is isometric to the lattice M: indeed, let

$$P := \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix},$$

then, $\det(P) = -1$ and

$$P \begin{pmatrix} 4 & 9 \\ 9 & 18 \end{pmatrix} {}^tP = \begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}.$$

Remark 8 The non-toric smooth Fano 3-fold X'' does not admit a small toric degeneration [6].

Remark 9 The Clifford dimension, which is defined for a curve A as

$$\dim \mathrm{Cliff}(A) := \min \left\{ \deg \mathcal{L} - 2\left(h^0(\mathcal{L}) - 1\right) \,\middle|\, \mathcal{L} \in \mathrm{Pic}(A), h^0(\mathcal{L}), h^2(\mathcal{L}) \geq 2 \right\},$$

each for the line l, the plane cubic C, and the degree-nine curve K is

$$\dim \mathrm{Cliff}(l) = 1, \dim \mathrm{Cliff}(C) = 2, \dim \mathrm{Cliff}(K) = 3.$$

What we can find in [5] are : there exists a $K3$ surface that contains the curve K and this $K3$ surface also contains a line. Also, if a $K3$ surface consists of a curve of Clifford dimension 3, then, such curve must be the smooth irreducuble of the intersection of two cubic surfaces in $\boldsymbol{P}^3$. This example of $K3$ surfaces is given by Martens (see [5] and their references).

The smooth Fano 3-fold X'' obtained by blowing-up $\boldsymbol{P}^3$ along K contains $K3$ surfaces as its anticanonical members. Thus, we identify the Gorenstein $K3$ surfaces in $|-K_{X''}|$ and those in $|-K_{\boldsymbol{P}^3} - K|$ which are quartic surfaces in $\boldsymbol{P}^3$. As in [5], $K3$ surfaces in $|-K_{\boldsymbol{P}^3} - K|$ contains a line.

Let $\mathcal{F}_4$ be a family of Gorenstein $K3$ surfaces in X''.

Conjecture The family $\mathcal{F}_4$ of Gorenstein $K3$ surfaces is generically birationally corresponding to the families $\mathcal{F}_1, \mathcal{F}_2$.

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Makiko Mase,
e-mail: wmstar2@gmail.com
Department of Mathematics and Information Sciences,
Tokyo Metropolitan University, 192-0397, 1-1 Minami Osawa,
Hachioji-shi, Tokyo, Japan

UK-Japan Winter School 2012, University of Oxford,
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