Hilbert Series
SUSY Gauge Theories

Amihay Hanany

Introduction

- N=1 gauge theories in 3+1 dimensions
- Moduli Space of Vacua
- Superpotential
- Chiral Ring
- Witten Index
- Count Chiral Operators in the Chiral Ring

Motivation

- Better understanding of
  - SUSY Gauge Theories
  - String Backgrounds

Hilbert Series

- Partition function
- Counts Chiral Operators in the Chiral Ring

Example
Free Chiral Multiplet

- X chiral (complex valued)
- $1, X, X^2, X^3, X^4, \ldots$ chiral (holomorphic)
- Conserved charge - number of X's
- Fugacity t conjugate to charge

$$H(t) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$$

Example
Free Chiral Multiplet

- Moduli Space is C, of dimension 1
- H(t) has a simple pole at t=1
- dimension of the moduli space is the order of the pole at t=1
- chemical potential

$$t = e^{-\mu}$$

Example
n Free Chiral Multiplets

- X_i i=1..n chiral multiplets (complex valued)
- 1, X_i, $X_i X_j$, $X_i X_j X_k$, ... chiral (holomorphic)
- n conserved charges $U(1)^n$; global symmetry U(n)
- n fugacities t_i

$$H(t_1, \dots, t_n) = \prod_{i=1}^{n} \frac{1}{1-t_i} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} t_1^{k_1} \cdots t_n^{k_n}$$

n Free Chiral Multiplets Fugacity Map

- set $t_1 = t\, y_1$
- $t_2 = t\, y_2/y_1$
- $t_3 = t\, y_3/y_2$ ...
- $t_{n-1} = t\, y_{n-1}/y_{n-2}$
- $t_n = t/y_{n-1}$

Character Expansion

- t counts the number of X's
- y's SU(n) fugacities
- keep track of SU(n) weights
- set [k,0,...,0] to be the character of k-th rank symmetric representation of SU(n)

$$H(t, y_1 \dots, y_{n-1}) = \prod_{i=1}^{n} \frac{1}{1 - t_i} = \sum_{k=0}^{\infty} [k, 0, \dots, 0] t^k$$

Moduli Space
n Free Chiral Multiplet

- Moduli Space is C^n, of dimension n
- H(t,1,...,1) has a pole of order n at t=1
- dimension of the moduli space is the order of the pole at t=1

$$H(t, 1 \ldots, 1) = \frac{1}{(1-t)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} t^k$$

Character Expansion

- Hilbert Series admits an expansion in terms of the characters of the global symmetry

Hilbert Series
Gauge Theories

- Chiral Operators are gauge invariant
- Look for invariants of the gauge group
- transform under the global symmetry

SQCD
SU(2) 1 flavor (2 chiral)

- Moduli space is 1 dimensional, freely generated by a 1 invariant $M = Q_1 Q_2$

$$H(t_1 t_2) = \oint_{|z|=1} \frac{dz(1-z^2)}{2\pi i z} \prod_{i=1}^{2} \frac{1}{(1-t_i z)(1-t_i/z)} = \frac{1}{1-t_1 t_2}$$

SQCD
SU(2) 2 flavors

$$H(t, y_1, y_2, y_3) = \sum_{k=0}^{\infty} [0, k, 0]_{SU(4)} t^{2k}$$

- Moduli space is 5d hyper-surface, generated by 6 invariants at order 2, subject to 1 relation at order 4 $\quad M_{ij} = Q_i Q_j; \quad \mathrm{Pf} M = 0$

$$H(t, 1, 1, 1) = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)^2(k+3)}{12} t^{2k} = \frac{1-t^4}{(1-t^2)^6}$$

SQCD
SU(2) with n flavors

$$H(t, y_1, \ldots, y_{2n-1}) = \sum_{k=0}^{\infty} [0, k, 0, \ldots, 0]_{SU(2n)} t^{2k}$$

- At order 2k precisely one irrep with Young diagram of 2 x k boxes.

SQCD $N_f > N_c$
$SU(N_c)$ N_f flavors

$$H_{SU(N_c),N_f}(t,\tilde{t},\{y\},\{\tilde{y}\}) = \sum_{n_1,n_2,\ldots,n_{N_c-1},\ell,m\geq 0} [n_1,n_2,\ldots,n_{N_c-1},\ell,0,\ldots,0;0,\ldots,0,m,n_{N_c-1},\ldots,n_2,n_1]\ t^a\,\tilde{t}^b$$

- Global symmetry is $SU(N_f)_L$ x $SU(N_f)_R$
- N_c+1 dimensional positive cartesian lattice

$$a = \ell N_c + \sum_{j=1}^{k} j n_j; \quad b = m N_c + \sum_{j=1}^{k} j n_j$$

SQCD $N_f > N_c$
$Sp(N_c)$ N_f flavors

$$H_{Sp(N_c),N_f} = \sum_{n_2,n_4,\ldots,n_{2N_c}\geq 0} [0,n_2,0,n_4,0,n_6,0,\ldots,0,n_{2N_c},0,...,0]t^a$$

- Global symmetry is $SU(2N_f)$
- N_c dimensional positive cartesian lattice

$$a = \sum_{j=1}^{N_c} 2jn_{2j}$$

N=2 Quiver k SU(N) instantons on C^2

Figure 6: The $\mathcal{N} = 2$ quiver diagram for k $SU(N)$ instantons on $\mathbb{C}^2$. The circular node represents the $U(k)$ gauge symmetry and the square node represents the $SU(N)$ flavour symmetry. The line connecting the $SU(N)$ and $U(k)$ groups denotes kN bi-fundamental hypermultiplets, and the loop around the $U(k)$ group denotes the adjoint hypermultiplet.

$\mathcal{N}=1$ Quiver k SU(N) instantons on C^2

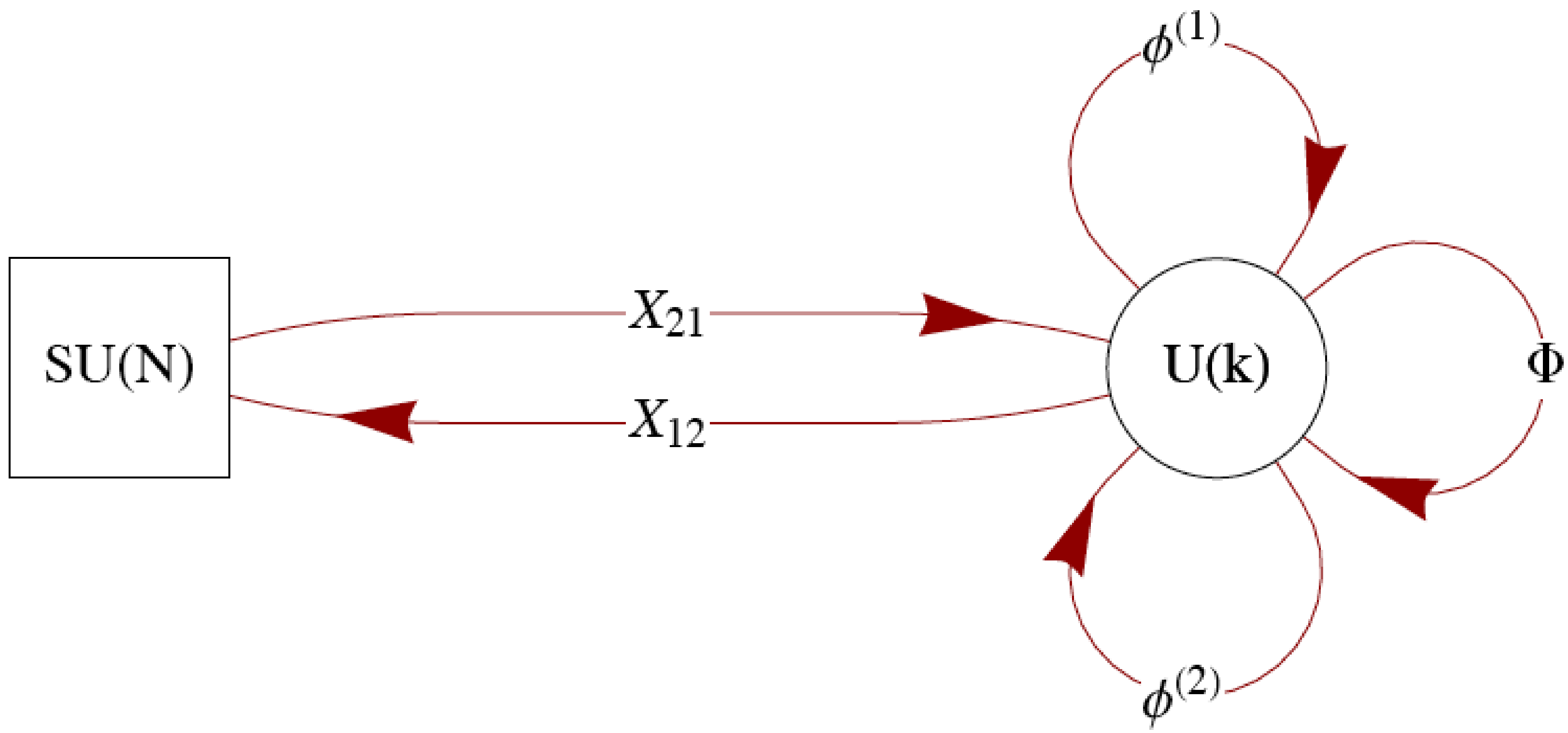


Figure 7: Flower quiver; The $\mathcal{N}=1$ quiver diagram for k $SU(N)$ instantons on $\mathbb{C}^2$ with the corresponding superpotential, $W = X_{21}\cdot\Phi\cdot X_{12} + \epsilon_{\alpha\beta}\phi^{(\alpha)}\cdot\Phi\cdot\phi^{(\beta)}$.

Moduli Space of 1 SU (N) instanton on C^2

- One dimensional lattice
- at order 2n: representation has highest weight equal to n times highest weight of adjoint
- Generalizes to any G instanton
- no ADHM for exceptional groups

$$H_{1,SU(N),\mathbb{C}^2} = \frac{1}{(1-tx)(1-\frac{t}{x})} \sum_{n=0}^{\infty} [n,0,\ldots,0,n] t^{2n}$$

Hilbert Series

- encodes information on:
- Chiral Ring
- Its generators, relations, dimension
- Exact function which characterizes the moduli space
- encoded as positive lattice of representations

Hilbert Series

- Palindromic <=> singular CY cone

Moduli Spaces

- Each moduli space has 3 numbers:
- Dimension D
- Number of generators G
- Number of relations R

Cases for the moduli space

- If R=0 : said to be freely generated
- If D=G-R : said to be a complete intersection
- Otherwise ...
- If G is exponential in D - wild moduli space
- If G is polynomial in D - tame moduli space

Thank you!

Notation

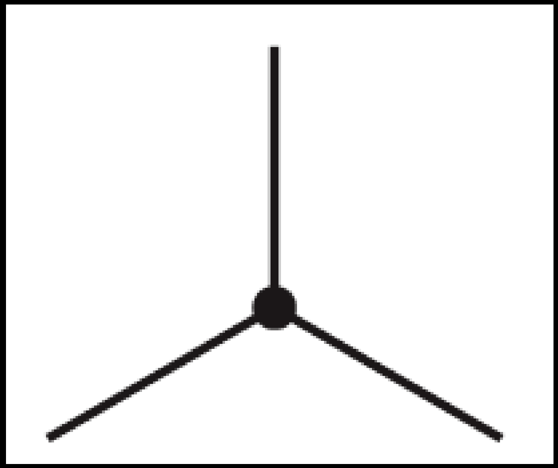
$$[n](x) = \sum_{m=-\frac{n}{2}}^{\frac{n}{2}} x^{2m} = \frac{x^{n+1} - x^{-n-1}}{x - x^{-1}}$$

Spin n/2 irrep of SU(2)

$$[n_1; \ldots; n_e] = \prod_{i=1}^{e} [n_i](x_i)$$

Example: g=0, e=3

$$g_{T_2}(t;x_1,x_2,x_3) = \mathrm{PE}\left[[1;1;1]t\right] = \prod_{\epsilon_i=\pm 1} \frac{1}{1 - t x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3}}$$

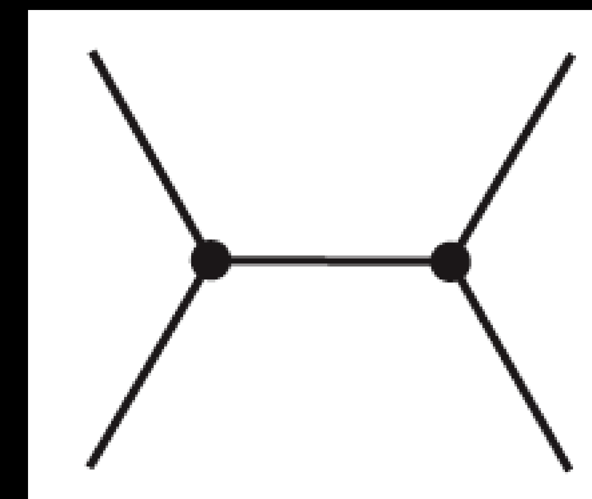


$$1 + [1;1;1]t + ([2;2;2] + [2;0;0] + [0;2;0] + [0;0;2])t^2$$

$$Q_{a,b,c} \qquad Q_{(a,b,c}Q_{a',b',c')} \qquad Q_{a,b,c}Q_{a',b',c'}\epsilon^{a,a'}\epsilon^{b,b'}$$

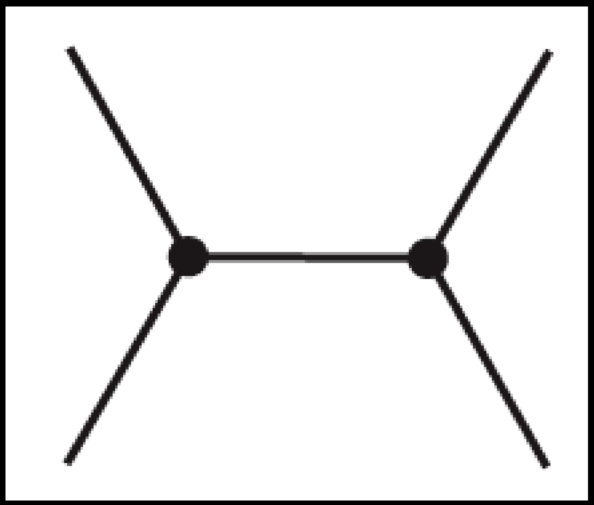
$$g_{T_2}(t;x_1,x_2,x_3) = \frac{1}{1-t^4} \sum_{n_1,n_2,n_3,m=0}^{\infty} \left([2n_1+m;2n_2+m;2n_3+m]t^{2n_1+2n_2+2n_3+m} + \right.$$
$$\left. [2n_1+m+1;2n_2+m+1;2n_3+m+1]t^{2n_1+2n_2+2n_3+m+3}\right) \quad .(4.3)$$

Example: g=0, e=4



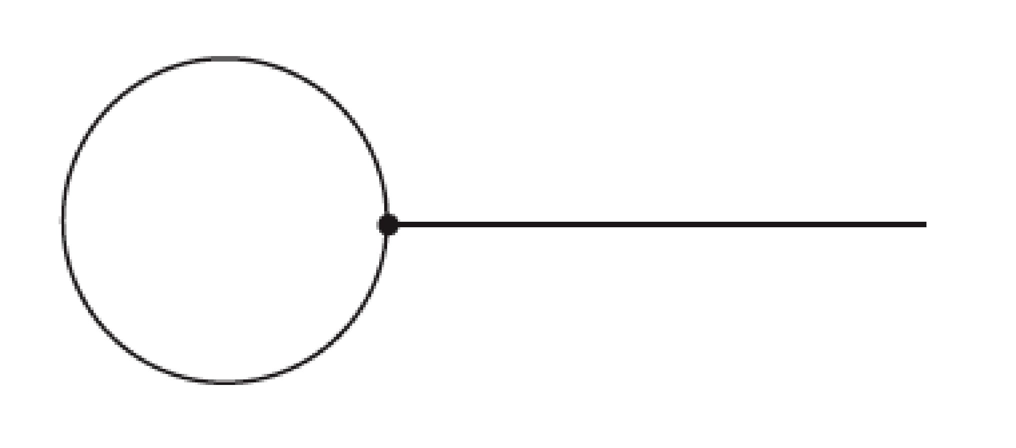
- To compute the HS observe
- Higgs branch is the moduli space of 1 SO (8) instanton on R^4
- HS was computed and gives
- $$g_{N_c=2,N_f=4}(t, z_1, z_2, z_3, z_4) = \sum_{k=0}^{\infty} [0, k, 0, 0]_{SO(8)} t^{2k}$$

HS: g=0, e=4



$$g_{N_c=2,N_f=4} = \frac{1}{1-t^4} \sum_{n_1,\ldots,n_4,m=0}^{\infty} \Big([2n_1+m; 2n_2+m; 2n_3+m; 2n_4+m] t^{2n_1+2n_2+2n_3+2n_4+2m} + [2n_1+m+1; 2n_2+m+1; 2n_3+m+1; 2n_4+m+1] t^{2n_1+2n_2+2n_3+2n_4+2m+4} \Big) \quad . \tag{4.7}$$

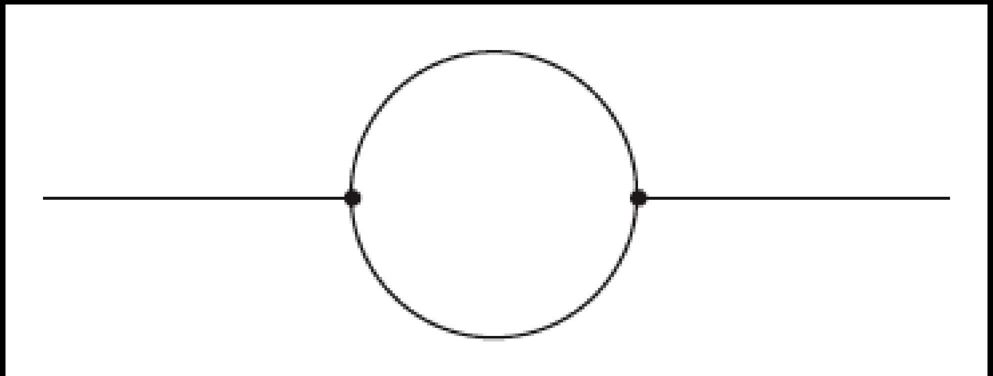
g=1, e=1 another method

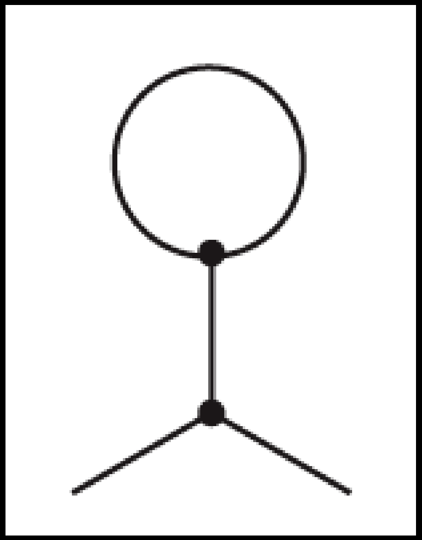


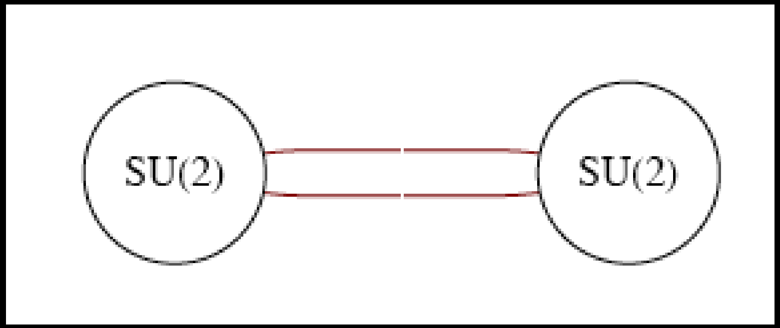
- two commuting adjoints
- symmetric product of 2 C^2 s

$$\begin{aligned} g_{\mathbb{C}^2/\mathbb{Z}_2\times\mathbb{C}^2}(t,x) &= \frac{1}{2}\left[\frac{1}{\left(1-\frac{t}{x}\right)(1-tx)}+\frac{1}{\left(1+\frac{t}{x}\right)(1+tx)}\right]\times\frac{1}{\left(1-\frac{t}{x}\right)(1-tx)} \\ &= (1-t^4)\mathrm{PE}\left[[1]t+[2]t^2\right] \ . \end{aligned} \quad (5.8)$$

Example: g=1, e=2

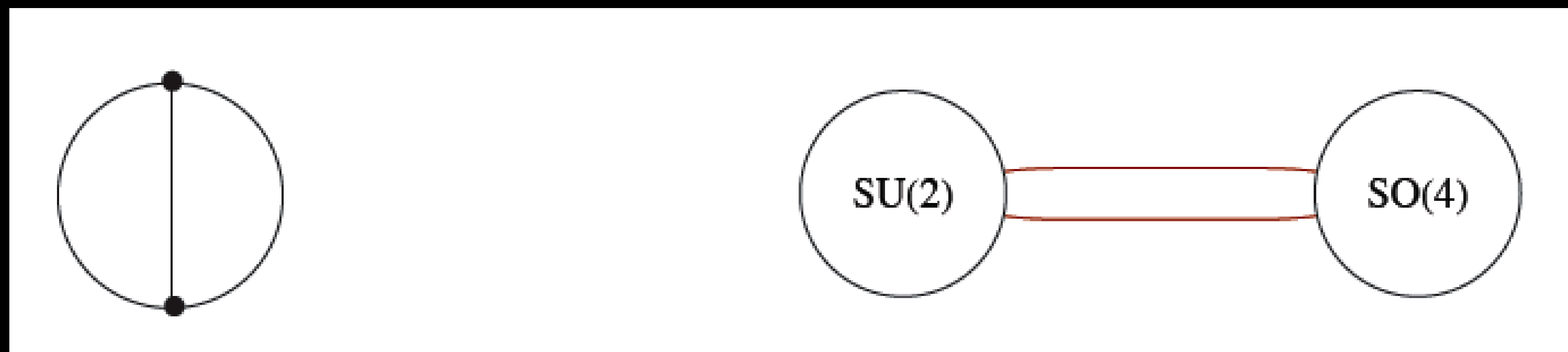






$$g_{A_1}(t, x_1, x_2) = \frac{1}{1-t^4} \sum_{n_1, n_2, m=0}^{\infty} [2n_1 + m; 2n_2 + m] t^{2n_1+2n_2+2m} + [2n_1 + m + 1; 2n_2 + m + 1] t^{2n_1+2n_2+2m+4} \ .$$

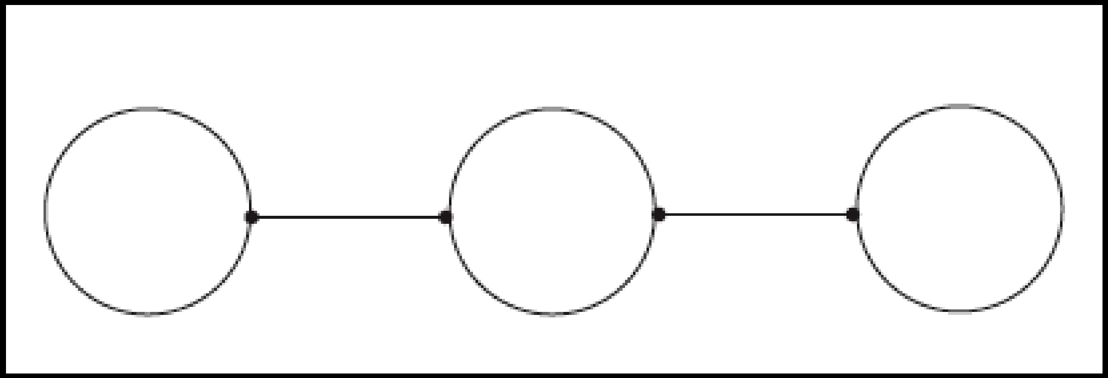
Example: g=2, e=0

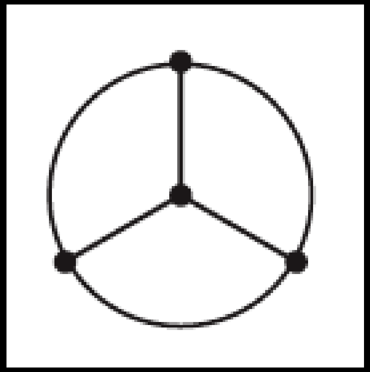


$\mathbb{C}^2/\hat{D}_3$

$$
\begin{aligned}
g_D(t) &= \int \mathrm{d}\mu_{SU(2)}(z) g_{\text{tadpole}}(t,z) g_{\text{glue}}(t,z) g_{\text{tadpole}}(t,z) \\
&= \oint_{|z|=1} \frac{\mathrm{d}z}{z}(1-z^2)\frac{(1-t^4)^2 \mathrm{PE}\left[2[1]_z t + 2[2]_z t^2\right]}{\mathrm{PE}\left[[2]_z t^2\right]} \\
&= \frac{1-t^8}{(1-t^2)(1-t^4)^2} \ .
\end{aligned}
$$

examples: g=3, e=0





general case: (g,e)

$$g_{(g,e)}(t, x_1, \ldots, x_e) = \frac{1}{1-t^4} \sum_{n_1=0}^{\infty} \cdots \sum_{n_e=0}^{\infty} \sum_{m=0}^{\infty} \Big([2n_1 + m, \ldots, 2n_e + m] \, t^{2n_1 + \ldots + 2n_e + \chi m} + [2n_1 + m + 1, \ldots, 2n_e + m + 1] \, t^{2n_1 + \ldots + 2n_e + \chi m + \chi + 2} \Big) \quad , \qquad (7.1)$$

$$f_m(t, x) \equiv \sum_{n=0}^{\infty} [2n + m]_x t^{2n} = (1 - t^2) \left([m]_x - [m-2]_x t^2 \right) \mathrm{PE} \left[[2] t^2 \right]$$

$$g_{(g,e)}(t, x_1, \ldots, x_e) = \frac{1}{1-t^4} \sum_{m=0}^{\infty} \left(t^{\chi m} \prod_{i=1}^{e} f_m(t, x_i) + t^{\chi m + \chi + 2} \prod_{i=1}^{e} f_{m+1}(t, x_i) \right)$$

generators

- 3e at dimension 2
- 2^e at dimension 2g-2+e
- Indication of the complexity at high e
- HyperKahler moduli space
- of dimension e+1
- with $SU(2)^e$ isometry