Exact results in Field Theories and Integrability

Nikolay Gromov

Part1

INTRODUCTION

Introduction – Quantum Mechanics

Free particle in 1D

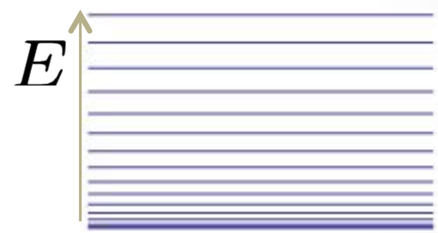
Classical $$x(t) = x(0) + \frac{p\,t}{m}$$

Energy: $E = \frac{p^2}{2m}$

Quantum $$\psi(x) = e^{ipx}$$

Particle on a circle: 

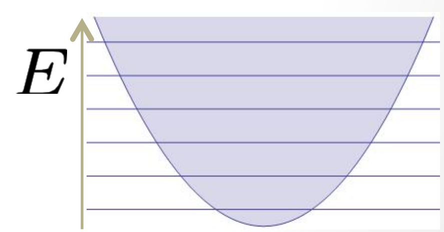
$$\psi(x+L) = \psi(x) \quad \Leftrightarrow \quad p = \frac{2\pi n}{L} \quad n \in \mathbb{Z}$$



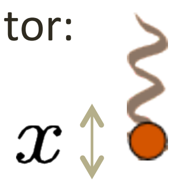
Particle in a potential

Schrödinger equation (e.v. problem for a linear diff operator):

$$-\frac{1}{2m}\partial^2\psi(x) + V(x)\psi(x) = E_n\psi(x)$$



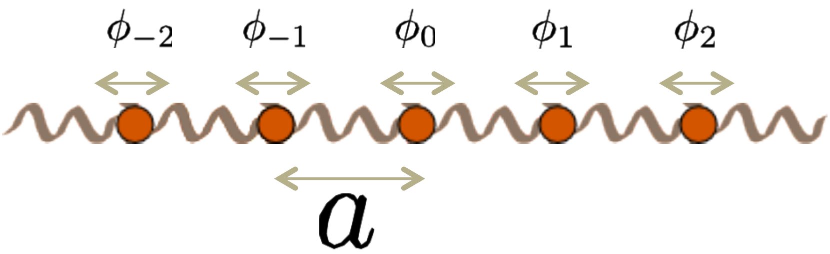
For harmonic oscillator:
$V(x) \sim x^2$



$$E_n = \omega(n + 1/2)$$

Introduction - Quantum Field Theory

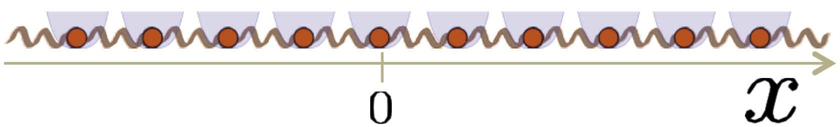
Infinite chain of oscillators:



Spectrum: infinite order diff. operator.
Wave function is a function of all
Coordinates:

$$\psi(\ldots, \phi_{-1}, \phi_0, \phi_1, \ldots)$$

Continuum limit:

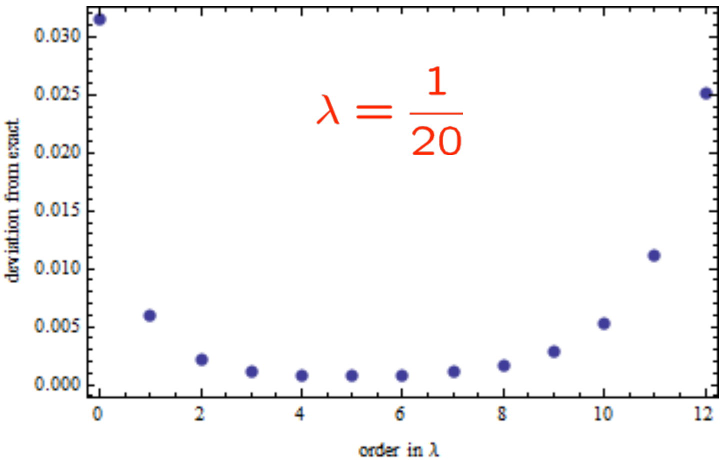


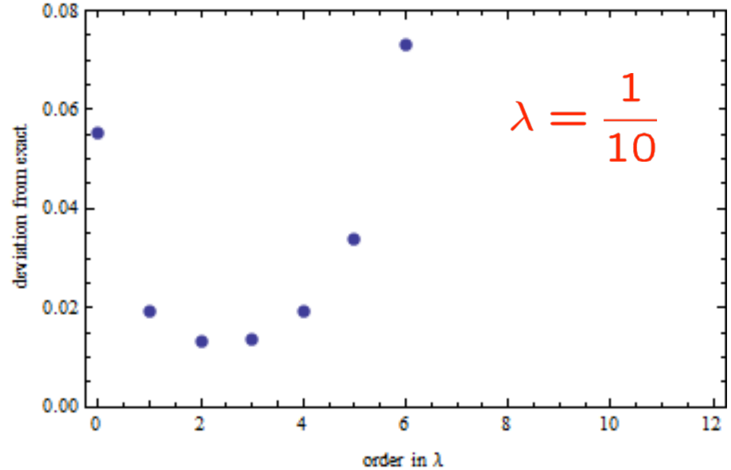
1) Define field $\phi(x)$: $\phi(na) \equiv \phi_a$
2) The classical equations of motion become: $\partial_t^2\phi(x,t) - \partial_x^2\phi(x,t) = 0$
3) Wave function become a functional $\psi[\phi(x)]$.
4) More transparent approach – path integral:

$$\int \mathcal{D}\phi \; e^{-S[\phi]} = \int \mathcal{D}\phi \; e^{-\int dtdx((\partial_t\phi)^2 + (\partial_x\phi)^2 + \mu\phi^2 + \lambda\phi^4)}$$

Introduction – Perturbation theory

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\phi e^{-\phi^2-\lambda\phi^4} \simeq \frac{1}{\sqrt{\pi}}\int d\phi e^{-\phi^2}(1-\lambda\phi^4+\ldots) = 1-\frac{3}{4}\lambda+\frac{105}{32}\lambda^2+\ldots$$





One example which works nicely – QED. The expansion parameter is $\alpha \sim 1/137$

Anomalous magnetic dipole moment g-2 is computed up to the order α^4.
(it takes more the 10years on numerical integration on supercomputers).
Experimentally: 0.00231930436171(52).
First order is simple: $\frac{\alpha}{\pi} \simeq 0.002323$

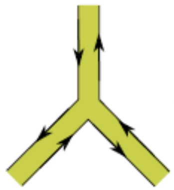
Gabrielse,Hanneke,Kinoshita,Nio,Odom 2006

Introduction – Planar Limit

Yang-Mills for SU(Nc) gauge group:

$$S=\frac{1}{4\,g_{\mathrm{YM}}^{2}}\int \mathrm{d}^{4}x\ \mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right), \quad F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}+[A_{\mu},A_{\nu}], \quad (A_{\mu})_{ij}=A_{\mu}^{a}\,(T^{a})_{ij}$$

$\sim g_{\mathrm{YM}}^{2}$



$\sim \frac{1}{g_{\mathrm{YM}}^{2}}$

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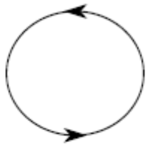
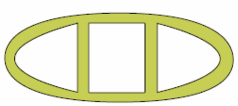


$\sim N_c$

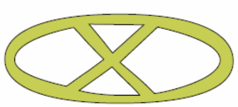
't Hooft coupling $\lambda = g_{YM}^{2} N_c$

1) Expand Nc is large 2) each order in Nc in perturbation theory in λ

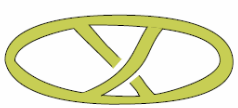
$\sim (g_{\mathrm{YM}}^{2})^{3-2}\,N_c^{3}=\lambda\,N_c^{2}$



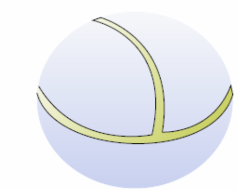
$\sim (g_{\mathrm{YM}}^{2})^{6-4}\,N_c^{4}=\lambda^{2}\,N_c^{2}$

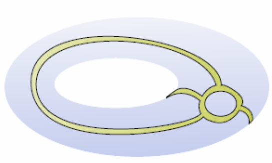


$\sim (g_{\mathrm{YM}}^{2})^{8-5}\,N_c^{5}=\lambda^{3}\,N_c^{2}$



$\sim (g_{\mathrm{YM}}^{2})^{6-4}\,N_c^{2}=\lambda^{2}$

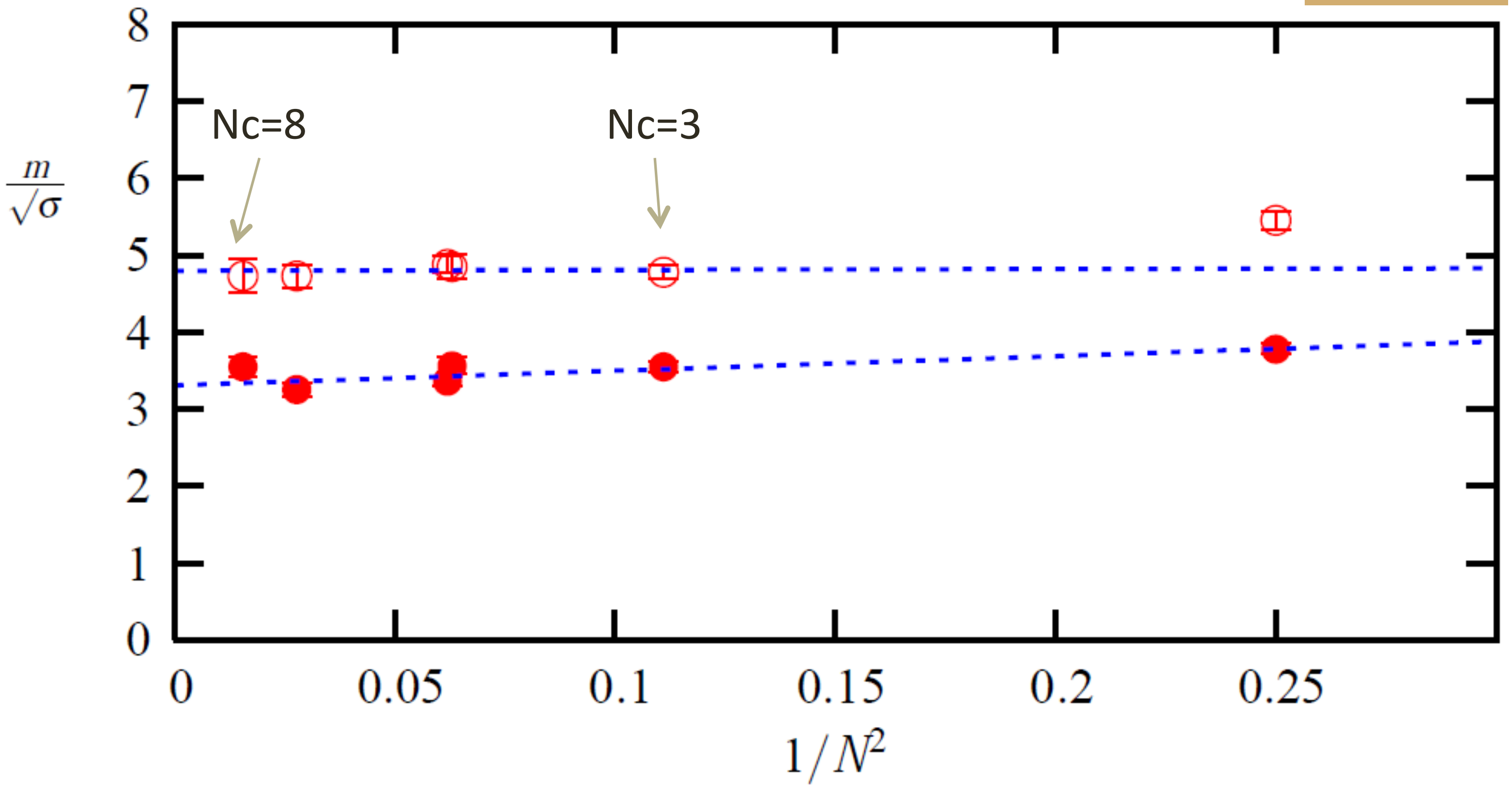




Introduction - Planar limit

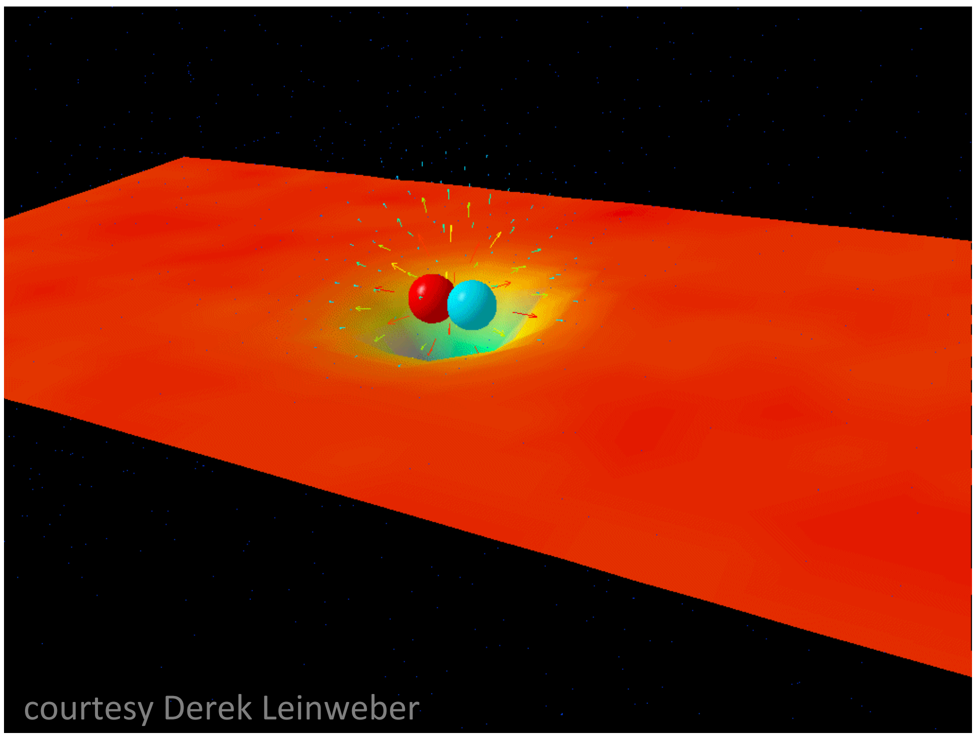
Expansion in large $\mathbf{N_c}$ organized in $\mathbf{1/N_c^2}$ series. For real QCD $\mathbf{1/N_c^2=1/9}$.

Teper 2008

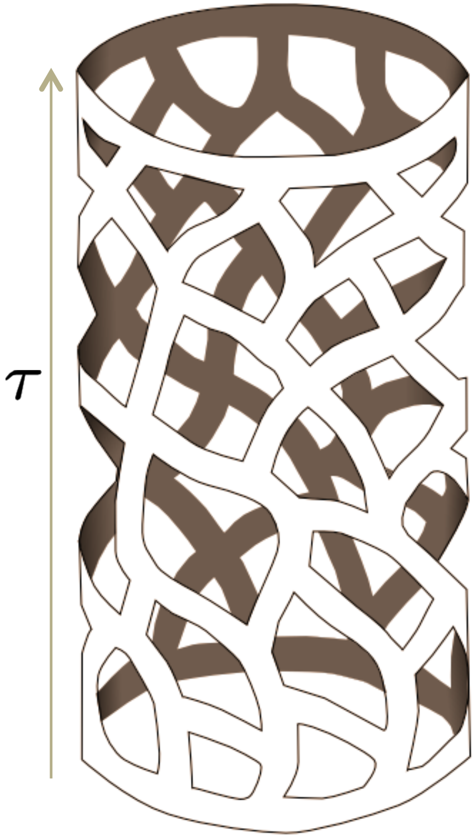


Introduction – Guage/String duality

2) Effective description of QCD by a string theory



1) ‘t Hooft limit of Feyman diagrams



Toy model - N=4 SYM

The "simplest" generalization of QCD:

$$S = \frac{1}{4g_{YM}^2}\int d^4x \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu} + \ldots)$$

Plus extra scalar fields $\Phi_1, \ldots, \Phi_6$ and fermions

Parameters: $\lambda = g_{YM}^2 N_c$ and $N_c = \infty$

Symmetries:

Lorentz: $so(3,1)$

$\Downarrow$

Conformal: $so(4,2) \Updownarrow su(2,2)$ $\quad so(6) \Updownarrow su(4)$ - rotation of the scalars

Sohnius, West 1981

Super (graded) Lie algebra:

$$\underbrace{su(2,2) \quad su(4)}_{su(2,2|4)}$$

$$su(2,2|4)$$

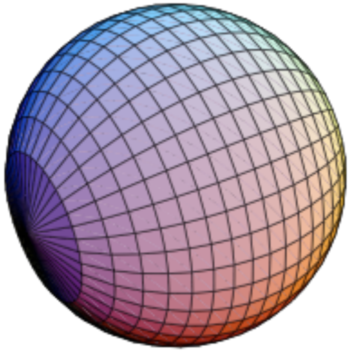
$$\mathbb{R}^3 \times \mathbb{R} \to \mathbb{S}^3 \times \mathbb{R}$$

Precise formulation - N=4 SYM

Maldacena

String in AdS_5xS^5

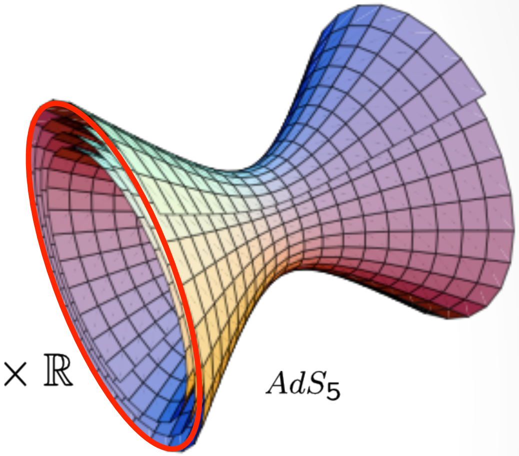
$$X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 = 1$$



S^5

x

$$Y_0^2 + Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 - Y_5^2 = 1$$



$\mathbb{S}^3 \times \mathbb{R}$

AdS_5

Symmetries:

$$\underbrace{so(6) \qquad \qquad so(4,2)}_{su(2,2|4)}$$

Paramiter:

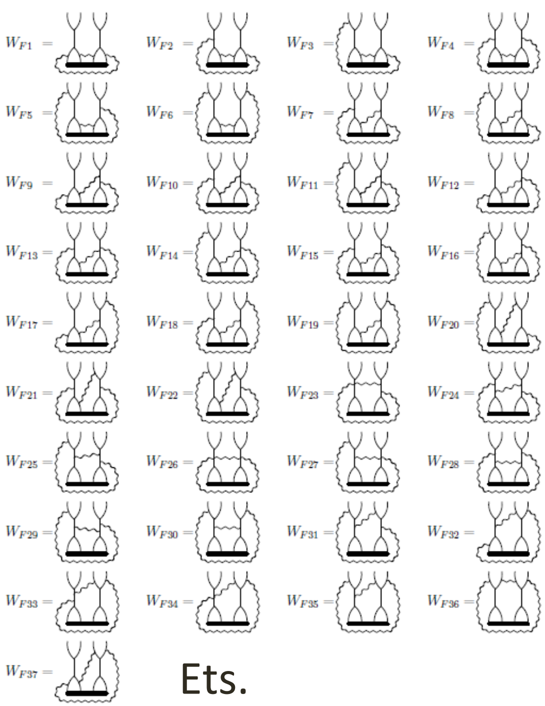
String tension: $T = \frac{\sqrt{\lambda}}{2\pi}$

Toy model - N=4 SYM

Still it is very nontrivial interacting theory. The simplest observable

$$\left\langle \mathrm{Tr}\left(\sum_{a=1}^{6}\Phi_a(x)\Phi_a(x)\right)\mathrm{Tr}\left(\sum_{a=1}^{6}\Phi_a(y)\Phi_a(y)\right)\right\rangle = \frac{1}{(x-y)^{2\Delta(\lambda)}}$$

Is known in perturbation theory up to 4-loops:



Ets.

Konishi dimention:

$$\Delta(\lambda) = 2+12\lambda-48\lambda^2+336\lambda^3+(-2496+576\zeta_3-1440\zeta_5)\lambda^4$$

Fiamberti, Santambrogio, Sieg, Zanon

Part2

GOAL

Goal

- Solve the simplest 4D gauge theory

Tools:

- Gauge/String duality

- Integrability

Part3

INTEGRABILITY

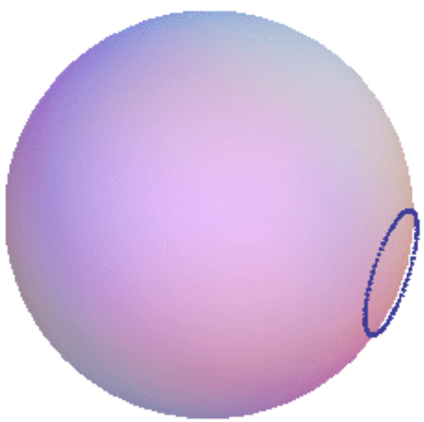
Classical Integrability

$$S = \int d\tau d\sigma \sum_{a=0}^{4} \left((\partial_\tau X_a)^2 - (\partial_\sigma X_a)^2 \right)$$

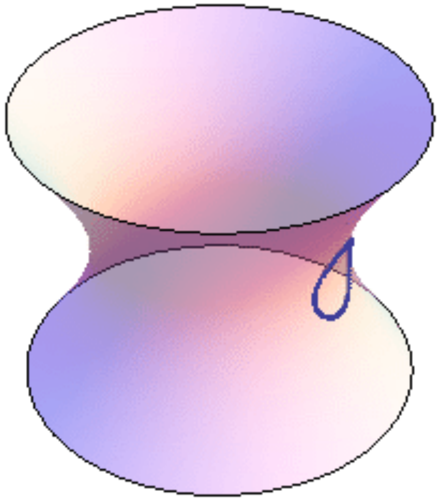
The scalar fields are constrained

$$X_a^2 = 1$$

$$\partial_\mu \partial^\mu X_a + (\partial_\nu X_b \partial^\nu X_b) X_a = 0$$



x



Integrability: infinitely many integrals of motion!

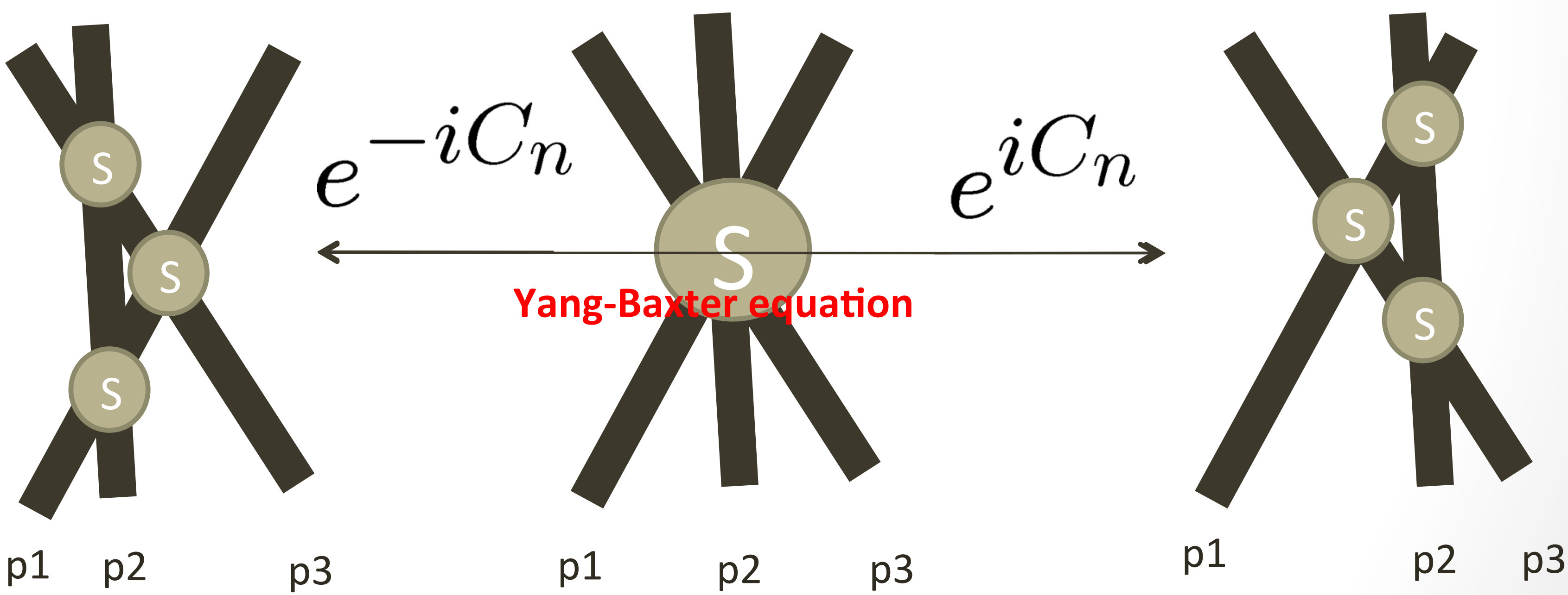
Quantum Integrability

The outgoing momenta are constrained:

$$A_n = \sum_i \omega_n(k_i) = \sum_i \omega_n(p_i) \quad , \quad n = 1, \ldots$$

The only solution: $m = m' \quad \{k_i\} = \{p_i\}$

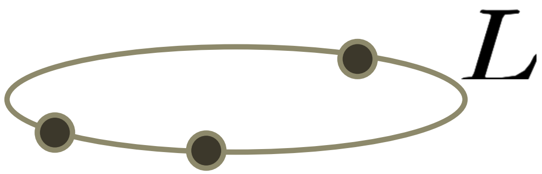
i.e. no creation or annihilation of particles



Asymptotic spectrum

- We can derive the momentum quantization condition

Particles on a ring:



$$\Psi(x_1+L, x_2, \ldots) = e^{ip_1 L} S(p_1, p_2) \ldots S(p_1, p_n) \Psi(x_1, x_2, \ldots)$$

- From periodicity of the wave function

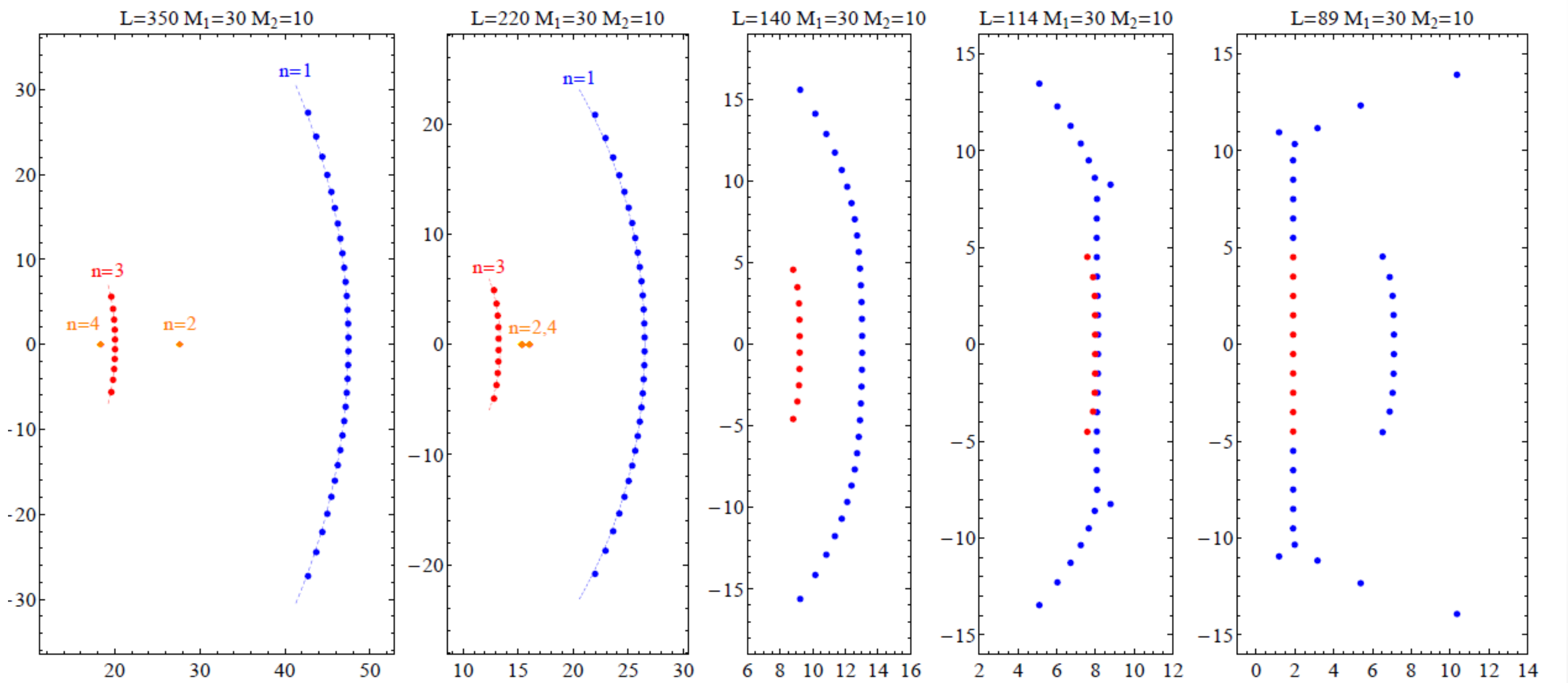
$$e^{ip_i L} = \prod_{j=1}^{M} S(p_i, p_j)$$

(Before: $\psi(x+L) = \psi(x) \;\Leftrightarrow\; p = \frac{2\pi n}{L} \;\; n \in \mathbb{Z}$)

Bethe ansatz - example

Heisenberg spin chain (just SU(2) symmetry group):

$$e^{ip_k L} = \prod_j^M \frac{u_k - u_j - i/2}{u_k - u_j + i/2} \;, \quad u_k \equiv \frac{1}{2}\cot\frac{p_k}{2} \qquad E = \sum_j \cos\frac{p_j}{2}$$

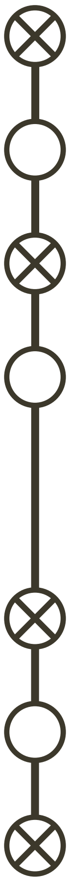


Till Bargheer, Niklas Beisert, N. G.

Beisert, Staudacher;
Beisert, Hernandez, Lopez;
Beisert,Eden,Staudacher

Asymptotic spectrum

Beisert, Staudacher;
Beisert, Hernandez, Lopez;
Beisert,Eden,Staudacher



$$1=\prod_{j=1}^{K_2}\frac{u_{1,k}-u_{2,j}+\frac{i}{2}}{u_{1,k}-u_{2,j}-\frac{i}{2}}\prod_{j=1}^{K_4}\frac{1-1/x_{1,k}x^+_{4,j}}{1-1/x_{1,k}x^-_{4,j}},$$

$$1=\prod_{j\neq k}^{K_2}\frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i}\prod_{j=1}^{K_3}\frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}}\prod_{j=1}^{K_1}\frac{u_{2,k}-u_{1,j}+\frac{i}{2}}{u_{2,k}-u_{1,j}-\frac{i}{2}},$$

$$1=\prod_{j=1}^{K_2}\frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}}\prod_{j=1}^{K_4}\frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}},$$

$$1=\left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^L\prod_{j\neq k}^{K_4}\frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\prod_{j}^{K_4}\left(\frac{1-1/x^+_{4,k}x^-_{4,j}}{1-1/x^-_{4,k}x^+_{4,j}}\right)\sigma^2(x_{4,k},x_{4,j})$$

$$\times\prod_{j=1}^{K_1}\frac{1-1/x^-_{4,k}x_{1,j}}{1-1/x^+_{4,k}x_{1,j}}\prod_{j=1}^{K_3}\frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}}\prod_{j=1}^{K_5}\frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}}\prod_{j=1}^{K_7}\frac{1-1/x^-_{4,k}x_{7,j}}{1-1/x^+_{4,k}x_{7,j}},$$

$$1=\prod_{j=1}^{K_6}\frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}}\prod_{j=1}^{K_4}\frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}},$$

$$1=\prod_{j\neq k}^{K_6}\frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i}\prod_{j=1}^{K_5}\frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}}\prod_{j=1}^{K_7}\frac{u_{6,k}-u_{7,j}+\frac{i}{2}}{u_{6,k}-u_{7,j}-\frac{i}{2}},$$

$$1=\prod_{j=1}^{K_6}\frac{u_{7,k}-u_{6,j}+\frac{i}{2}}{u_{7,k}-u_{6,j}-\frac{i}{2}}\prod_{j=1}^{K_4}\frac{1-1/x_{7,k}x^+_{4,j}}{1-1/x_{7,k}x^-_{4,j}}.$$

$$x+\frac{1}{x}=\frac{u}{g}\ ,\quad x^{\pm}+\frac{1}{x^{\pm}}=\frac{u\pm i/2}{g}\qquad E=\sum_k\epsilon_k=\sum_k 2gi\left(\frac{1}{x^+_{4,k}}-\frac{1}{x^-_{4,k}}\right)$$

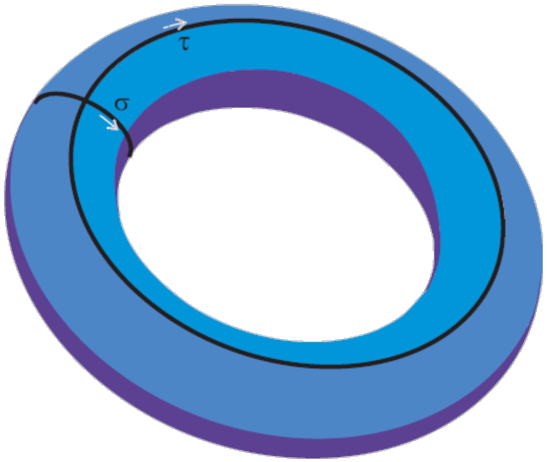
Finite volume spectrum

For Konishi operator L=2 (the one with $\Delta(\lambda) = 2+12\lambda-48\lambda^2+336\lambda^3+(-2496+576\zeta_3-1440\zeta_5)\lambda^4$).

Partition function of a relativistic model obeys:

$$Z(L,\beta) = \sum_n e^{-E_n(L)\beta} = \sum_n e^{-E_n(\beta)L}$$

Since it is given by a Path integral on the torus:



…,Matsubara, Zamolodchikov,…

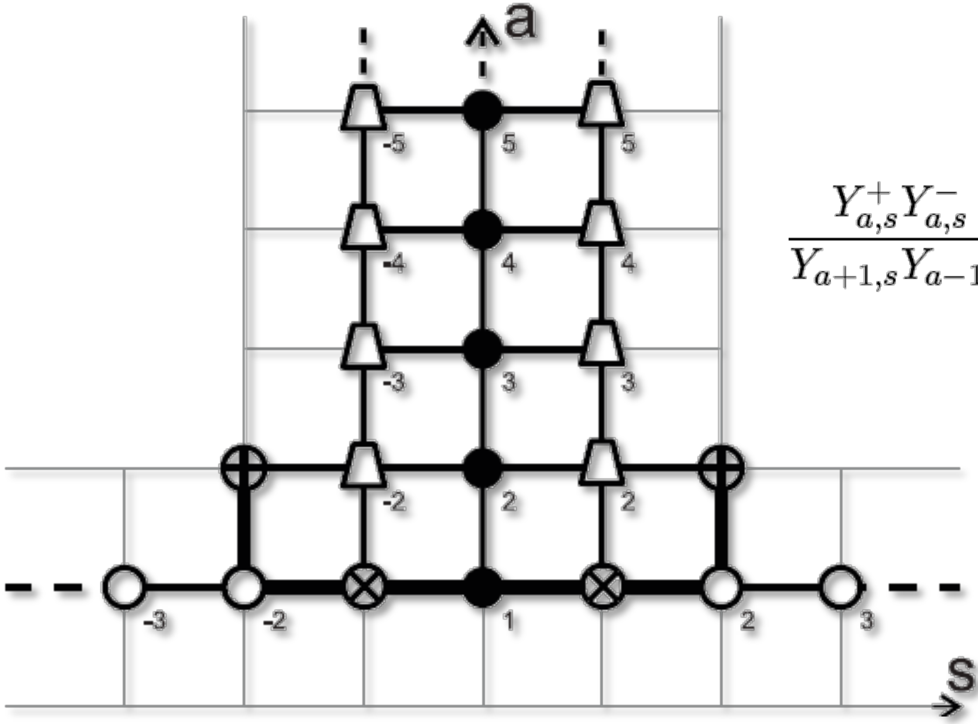
$$\sum e^{-E_n(L)R} = \sum e^{-E_n(R)L}$$

$$\Downarrow$$

$$e^{-E_0(L)R}$$

I.e. from the asymptotical spectrum (infinite R) we can compute the Ground state energy for ANY finite volume!

Y-system



N.G., Kazakov, Vieira

$$\frac{Y_{a,s}^{+}Y_{a,s}^{-}}{Y_{a+1,s}Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

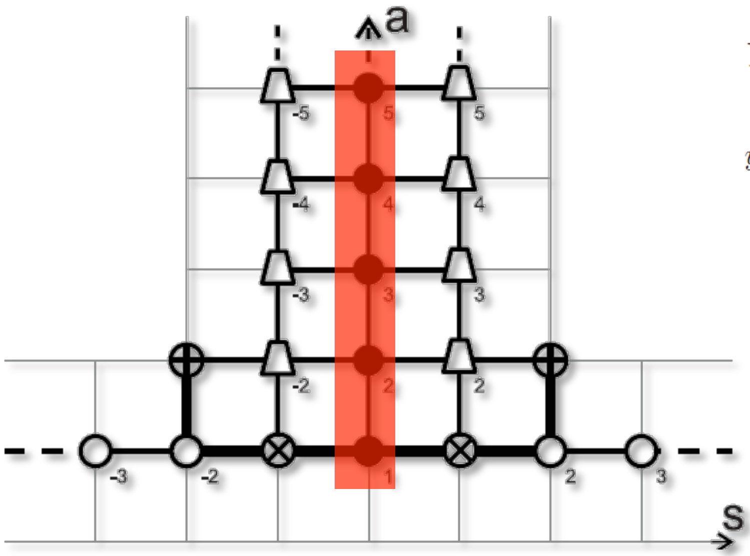
$$E = \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \frac{\partial \epsilon_a}{\partial u} \log\left(1 + Y_{a,0}(u)\right)$$

$$Y_{1,0}(u_{4,j}) = -1$$

Part4

RESULTS

Small λ



$$Y^*_{a,0} = g^8 \left(3\,2^7 \frac{3a^3+12au^2-4a}{(a^2+4u^2)^2}\right)^2 \frac{1}{y_a(u)y_{-a}(u)}$$

$$y_a(u) = 9a^4 - 36a^3 + 72u^2a^2 + 60a^2 - 144u^2a - 48a + 144u^4 + 48u^2 + 16$$

$$E = 2i \sum_{a=1}^{\infty} \int Y^*_{a,0} du$$

Reproduce!

$$\Delta(\lambda) = 2+12\lambda-48\lambda^2+336\lambda^3+(-2496+576\zeta_3-1440\zeta_5)\lambda^4$$

Sieg, Torrielli;
Janik, Bojnok,; Velizhanin
N.G., Kazakov, Vieira

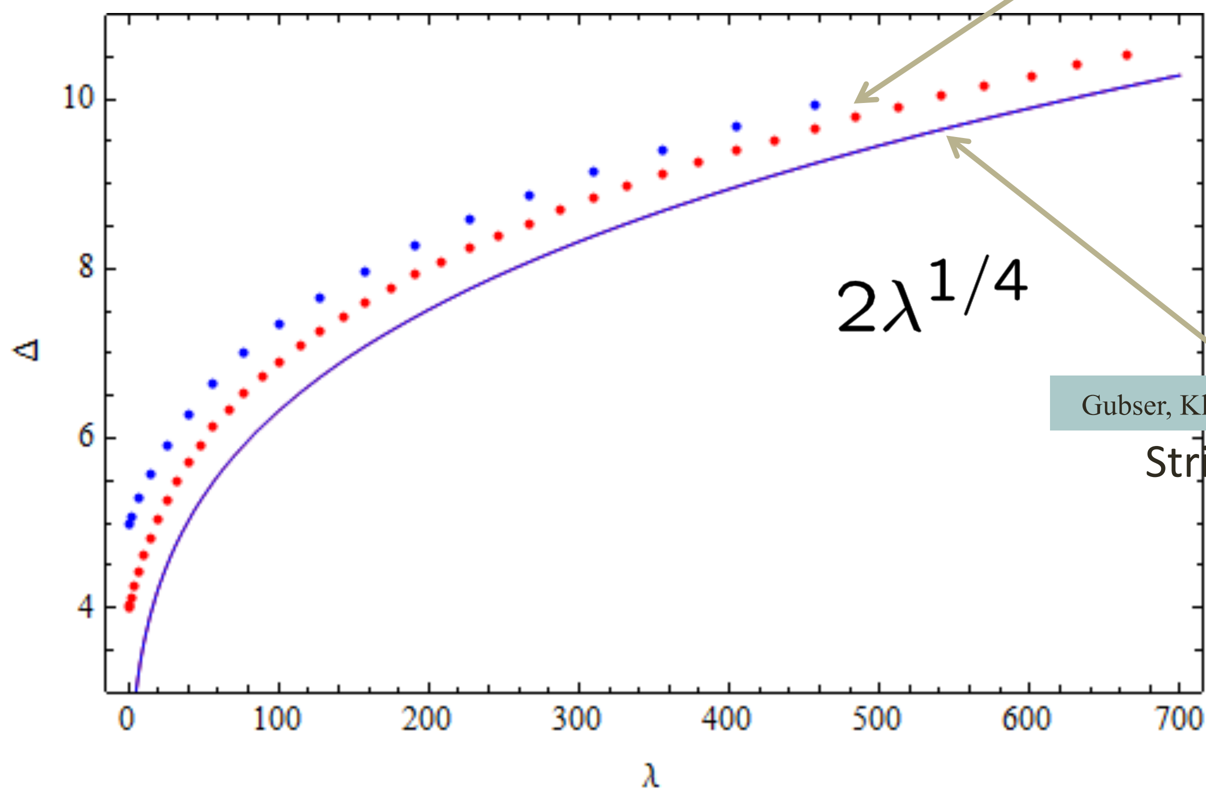
Intermediate λ

Cavaglià, Fioravanti, Tateo`10

Functional Y-system + analyticity conditions gives integral eqs.

N.G., Kazakov, Vieira '09

Integrability



Gubser, Klebanov, Polyakov `98

String theory

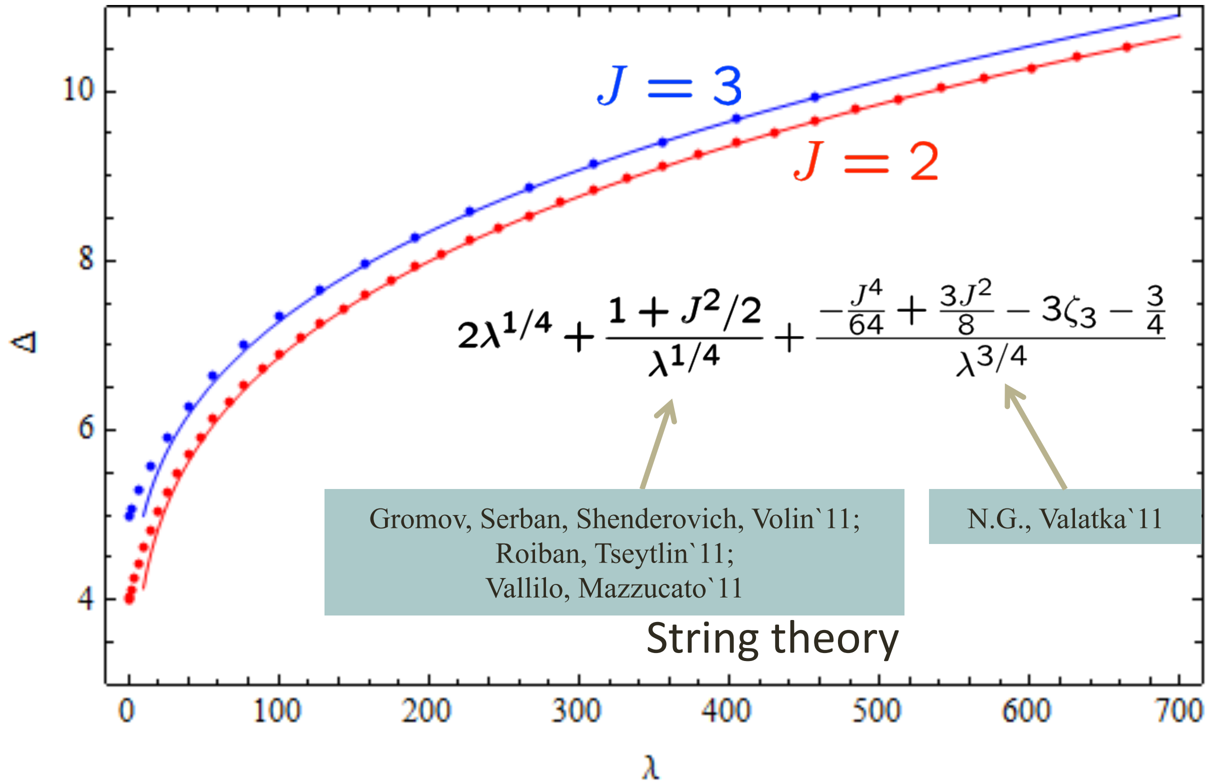
The integral equations can be also derived using TBA

Bombardelli, Fioravanti, Tateo`09
N.G., Kazakov, Vieira`09
Arutynov, Frolov`09

Intermediate λ

N.G., Kazakov, Vieira ‘09

Integrability



Science

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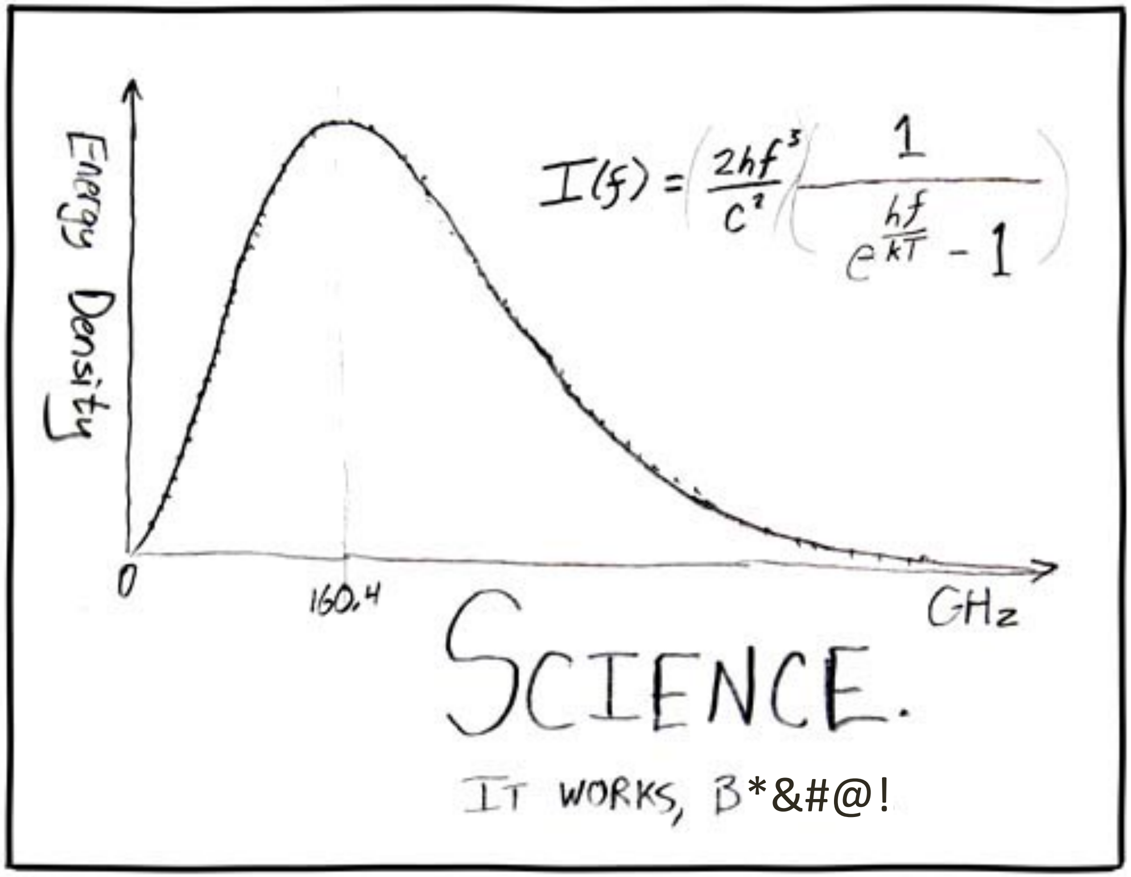

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Permanent link to this comic: http://xkcd.com/54/

Image URL (for hotlinking/embedding): http://imgs.xkcd.com/comics/science.jpg

Large λ analytically

N.G. 2009
N.G., V.Kazakov, Z.Tsuboi 2010

Answer is given in terms of the psu(2,2|4) characters (generalization of Schur polynomials):

$$T_{a,s} = \begin{cases} (-1)^{(a+1)s}\left(\dfrac{x_3x_4}{y_1y_2y_3y_4}\right)^{s-a} \dfrac{\det\left(S_i^{\theta_{j,s+2}}\, y_i^{\,j-4-(a+2)\theta_{j,s+2}}\right)_{1\le i,j\le 4}}{\det\left(S_i^{\theta_{j,0+2}}\, y_i^{\,j-4-(0+2)\theta_{j,0+2}}\right)_{1\le i,j\le 4}} & , \quad a \ge |s| \\[2ex] \dfrac{\det\left(Z_i^{(1-\theta_{j,a})}\, x_i^{\,2-j+(s-2)(1-\theta_{ja})}\right)_{1\le i,j\le 2}}{\det\left(Z_i^{(1-\theta_{j,0})}\, x_i^{\,2-j+(0-2)(1-\theta_{j,0})}\right)_{1\le i,j\le 2}} & , \quad s \ge +a \end{cases}$$

Hirota equation:

$$T_{a,s}^2 = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}$$

$$Y_{a,s} = \frac{T_{a+1,s}T_{a-1,s}}{T_{a,s+1}T_{a,s-1}}$$

Part5

GENERALIZATIONS

Less super-symmetries (N=1)

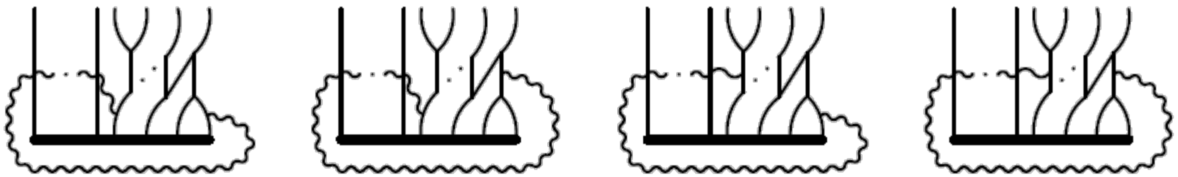
(Marginal deformations of N=4 SYM theory)

N.G., F.Levkovich-Maslyuk 2010

The integrability structure is the same. Only the boundary conditions are different.
Interesting – less degenerate more parameters.

Agrees with perturbation theory at least up to 11 loops:

Fiamberti, Santambrogio, Sieg, Zanon `08



We derive the generating function for these groups of Feynman integrals:

$$G_a(q) = \bar{q}^{L-1} \frac{a-1}{a(q^2-1)+2} \; {}_2\tilde{F}_1 \left[\frac{1}{2}, 1; \frac{3}{2} - L; \left(1 + \frac{2}{a(q^2-1)}\right)^{-2} \right]$$

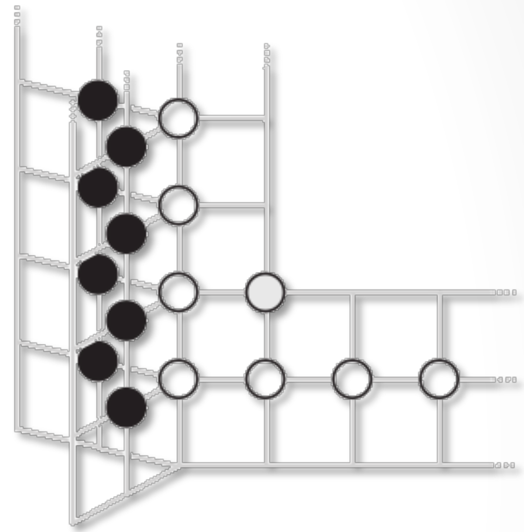
3D Gauge theory - ABJM

1) Bethe ansatz is known and well tested:

N.G., P.Vieira

2) Y-system is known:

N.G., V.Kazakov, P.Vieira 2009,
D.Bombardelli, D.Fioravanty,R.Tateo 2010
N.G., F.Levkovich-Maslyuk 2010



Prediction for some simples state:

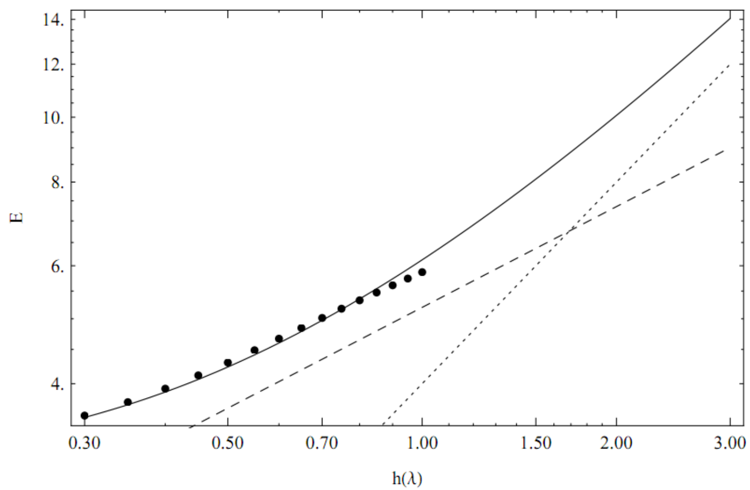
$$E_{\text{wrapping}} = 32 - 16\zeta(2)$$

N.G., V.Kazakov,P.Vieira`0901

Confirmed later by a direct 4-loop calculation:

Minahan, Sax, Sieg`0912

Recently Y-system was solved numerically:



CONCLUSIONS

- We propose the system of functional equations for the spectral problem of Y-system
- Passes all possible tests so far
- Hidden structures in the perturbation theory?
- Simplify integral equations
- What's next? -- 3-point correlators