Correlation Functions, Wilson Loops, Local Operators in Twistor Theory!

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We've seen loads of motivation for why these are interesting topics; studying gauge theory in the context of AdS/CFT or otherwise motivates a number of conjectures:

- Amplitude/Wilson Loop Duality: $A^n = \langle W^n[C] \rangle$
- Correlation Function/Wilson Loop Correspondence:

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle}{\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle^{\text{tree}}} = \langle W^n[C] \rangle^2$$

- Correlation Functions with Local Operators:

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n)\, \mathcal{O}(y) \rangle}{\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle} = \frac{\langle W^n[C]\, \mathcal{O}(y) \rangle}{\langle W^n[C] \rangle}$$

Loads of evidence for why these are true from working at strong and weak coupling, but can we find analytic proofs?

In this talk, we'll explore how to approach this problem using twistor theory!

Note: the gauge theory we'll consider in this talk is $\mathcal{N}=4$ SYM... the simplest 4-D gauge theory available to us!

Twistor Theory Toolbox:

$\mathbb{PT}$ is a suitable open subset of the Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$.

- Has vanishing first super-Chern class, possesses Ricci super-flat Kähler metric, the Berezinian sheaf has a canonical global section.
- $\mathbb{CP}^3$, $\mathcal{O}\left(\bigoplus_{i=0}^{4} \Lambda^i \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 4}\right)$

Work with homogeneous coordinates $Z^I = (Z^\alpha, \chi^a) = (\lambda_A, \mu^{A'}, \chi^a)$, where λ, μ are 2-component Weyl spinors and χ^a $(a=1,\dots,4)$ are Grassmann/fermionic coordinates.

The natural space-time for $\mathcal{N}=4$ SYM is chiral Minkowski superspace, which has coordinates (x^μ, θ^{Aa}), or $(x^{AA'}, \theta^{Aa})$, where $x^{AA'} = \sigma_\mu^{AA'} x^\mu$

$\mathbb{PT}$ is related to $\mathbb{M}$ by the twistor incidence relations:

$$\begin{cases} \mu^{A'} = i x^{AA'} \lambda_A \\ \chi^a = \theta^{Aa} \lambda_A \end{cases}$$

These give an equation for a linearly embedded $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^{3|4}$.

Hence, we have $(x,\theta) \in \mathbb{M} \longleftrightarrow X \cong \mathbb{CP}^1 \subset \mathbb{PT}$. Further, X_1, X_2 intersect iff (x_1, x_2) are null separated from each other (i.e., $(x_1 - x_2)^{AA'} = \lambda^A \tilde{\lambda}^{A'}$, $(\theta_1 - \theta_2)^{Aa} = \lambda^A \eta^a$)

Some important results in twistor theory which I won't prove:

- <u>Penrose Transform</u>: Let $\mathcal{U} \subset \mathbb{CP}^3$ be an open subset, and $\mathcal{O}(n)$ the sheaf of holomorphic functions homogeneous of degree n. Then
  $H^1(\mathcal{U}, \mathcal{O}(2h-2)) \cong \{\text{z.r.m. fields on } \mathbb{M}_b \text{ of helicity } h\}$.
- <u>Ward Correspondence</u>: there is a one-to-one equivalence between Yang-Mills instantons $(G = SU(N))$ on $\mathbb{M}_b$ and holomorphic rank-N bundles $E \to \mathbb{PT}_b$ such that $E|_X$ is trivial, $\det E$ is trivial, and E admits a positive real form.

The Penrose transform can be realized easily using integral formulae.

<u>E.g.</u> let $\phi \in H^1(\mathcal{U}, \mathcal{O}(-2))$. Then $\Phi(x) = \int_X \langle \lambda d\lambda \rangle \wedge \phi|_X$ obeys $\Box \Phi = 0$.

similarly, for $\psi \in H^1(\mathcal{U}, \mathcal{O}(-3))$, $\psi_A(x) = \int_X \langle \lambda d\lambda \rangle \wedge \lambda_A \psi|_X$ obeys $\nabla^{AA'} \psi_A = 0$.

<u>Gauge Theory on twistor space</u>:

The Penrose transform gives us an easy way of encoding the field content of $\mathcal{N}=4$ SYM twistorially.

Setup: $G = SU(N)$, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_N$, $E \to \mathbb{PT}$ such that $c_1(E) = 0$ (or at least $c_1(E)|_X = 0$).

A $(0,1)$-connection on E can be given by specifying some $\mathcal{A} \in \Omega^{0,1}(\mathbb{PT}, \mathrm{End}(E))$, which has no form components in the fermionic directions.

Expanding in χ gives: $\mathcal{A} = a + \chi^a \tilde{\psi}_a + \frac{\chi^a \chi^b}{2} \phi_{ab} + \frac{\varepsilon_{abcd}}{3!} \chi^a \chi^b \chi^c \tilde{\psi}^d + \frac{\chi^4}{4!} g$. When $\bar{\partial} \mathcal{A} = 0$, this reproduces the $\mathcal{N}=4$ SYM multiplet via the Penrose transform.

But how do we formulate our QFT on $\mathbb{PT}$?

Recall that Yang-Mills theory has a chiral decomposition in 4-d (see Mat's talk).

$$S_{YM}[A] = \frac{1}{4g^2}\int_{\mathbb{M}} \mathrm{tr}(F \wedge * F) \longrightarrow \text{Field Eqns } \nabla * F = 0.$$

Now, introduce an auxiliary 2-form $G \in \Omega^{2,-}(\mathbb{M}, \mathfrak{sl}_N)$, and consider $S_{CS}[A,G] = \int_{\mathbb{M}} \mathrm{tr}(G \wedge F) + \lambda \int_{\mathbb{M}} \mathrm{tr}(G \wedge G)$.

This has field equations $F^- = \lambda G$, $\nabla G = 0$. Are these equivalent to the Yang-Mills equation? $\nabla * F = \nabla(F^+ - F^-) = \nabla(F - 2F^-) = \nabla F - 2\lambda \nabla G = 0$.

Hence, this chiral formalism is perturbatively equivalent to the Yang-Mills Lagrangian!

When $\lambda = 0$, get $F^- = 0 \Leftrightarrow$ Yang-Mills instantons. The second term represents the ASD interactions of the theory.

Same is true for $N=4$ SYM!

SD Sector: simple SUSY generalization of the Ward Correspondence tells us that instantons on $\mathbb{M} \longleftrightarrow$ holomorphic bundles on $\mathbb{PT}$. So we need $F^{0,2}(E \to \mathbb{PT}) = \bar{\partial}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$.

These are the field equations of holomorphic Chern-Simons theory, so take:

$$S_{SD}[\mathcal{A}] = \frac{i}{2\pi}\int_{\mathbb{PT}} D^{3|4}Z \wedge \mathrm{tr}\left(\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \tfrac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right).$$

ASD Interactions: motivations from twistor-string theory suggest:

$$I[\mathcal{A}] = \int_{\mathbb{M}_{\mathbb{R}}} d^{4|8}X \, \log\det(\bar{\partial} + \mathcal{A})|_X .$$

Does this integral make sense? $\det(\bar{\partial} + \mathcal{A})|_X \in \mathcal{L} \longrightarrow \mathrm{Con}(E \to \mathbb{PT})|_X \cong \mathrm{Con}(E \to \mathbb{P}^1)$

Have

$$\begin{array}{ccc} \mathcal{L} & & \\ \downarrow & & \\ \mathrm{Con}(E \to \mathbb{P}^1) & \longleftarrow & \mathrm{Con}(E \times \mathbb{P}^1) \times \overline{\mathcal{M}}_{0,1}(\mathbb{PT},1) = \mathrm{Con}(E \times \mathbb{P}^1) \times \mathbb{M} \end{array}$$

The Bismut-Freed index theorem then tells us that
$$F^{(2)} = \int_{\mathbb{M}} \mathrm{Td}(T\mathbb{M})\, ch(T\mathbb{M}\oplus E|_x) = 0,$$ so we can treat $\det(\bar\partial + \mathcal{A})$ as a "function" on $\mathbb{M}$.

So we have a twistor action: $S[\mathcal{A}] = S_{SD}[\mathcal{A}] + \lambda I[\mathcal{A}]$.

Facts: - gauge invariant under $(\bar\partial+\mathcal{A}) \to \gamma(\bar\partial+\mathcal{A})\gamma^{-1}$ for $\gamma \in \Gamma(E, \mathrm{End}(E))$
- perturbatively expanding $I[\mathcal{A}]$ gives the MHV vertices, objects which are also supported on lines in $\mathbb{PT}$.

Fixing gauge freedom:
- Woodhouse gauge: $\bar\partial^{\dagger}|_x \mathcal{A}|_x = 0$. Gauge transformations preserving this must obey $\bar\partial^{\dagger}\bar\partial|_x \gamma = 0 \Rightarrow \Delta|_x \gamma = 0$, so $\gamma = \gamma(x)$ and we recover space-time gauge freedom.
- CSW gauge: fix some $Z_* \in \mathbb{PT}$. This induces a foliation by lines through Z_*, and we demand that $\mathcal{A}$ vanish on the leaves of the foliation: $\overline{Z_*^I \frac{\partial}{\partial Z^I}} \lrcorner \mathcal{A} = 0$.

Theorem: Woodhouse gauge $\Longrightarrow$ Space-time $N=4$ SYM
CSW gauge $\Longrightarrow$ MHV formalism

From now on, we work in CSW gauge, where the propagator for the theory should be a Green's current for $\bar\partial$ and obey the CSW condition.
Find $\Delta_*(Z_1, Z_2) = \bar\delta^{2|4}(Z_1, Z_*, Z_2) \in D^{0,2}(\mathbb{PT})$ and obeys $\bar\partial\,\bar\delta^{2|4}(Z_1,*,Z_2) = \bar\delta_\Delta$.
So now we have a gauge theory on $\mathbb{PT}$; what about local operators and Wilson loops?

Local Operators:
We will work with the ½-BPS operators of $N=4$ SYM; these are single trace and protected (non-anomalous conformal dimension). This construction works for other operators, too!
Take $O(x) = O_{abab} = \mathrm{tr}(\Phi_{ab}\Phi_{ab})$. When $G = U(1)$, the Penrose transform tells us how to do this immediately: $$O^{U(1)}(x) = \int_{X\times X} \langle\lambda d\lambda\rangle\langle\lambda' d\lambda'\rangle \wedge \phi_{ab}(\lambda)\wedge\phi_{ab}(\lambda').$$

Easy SUSY generalization: $$O^{U(1)}(x,\theta) = \int_{X\times X} \langle\lambda d\lambda\rangle\langle\lambda' d\lambda'\rangle \wedge \frac{\partial^2 \mathcal{A}(\lambda)}{\partial\chi^a\partial\chi^b} \wedge \frac{\partial^2\mathcal{A}(\lambda')}{\partial\chi^a\partial\chi^b}$$

Here $\dfrac{\partial^2 \mathcal{A}}{\partial x^a \partial x^b} = \phi_{ab} + \varepsilon_{abcd} x^c \tilde{\psi}^d + \dfrac{\varepsilon_{abcd}}{2} x^c x^d g$.

This doesn't work for $G = SU(N)$ because we have no way of comparing the fibers of $E|_X$ at λ and λ'. To do this requires a holomorphic trivialization of $E|_X$.

Does such a thing exist? YES!

$\to$ $E|_X$ topologically trivial over $\mathbb{CP}^1$ $\Rightarrow$ $E|_X = \bigoplus_i \mathcal{O}(a_i)$ with $\sum_i a_i = 0$

$\to$ $E|_X$ holomorphic, $(\bar\partial + \mathcal{A})|_X^2 = 0$ $\Rightarrow$ $a_i = 0 \ \forall i$.

Hence, we can find a (unique) solution to the equation $\gamma(\bar\partial + \mathcal{A})|_X \gamma^{-1} = \bar\partial|_X$.

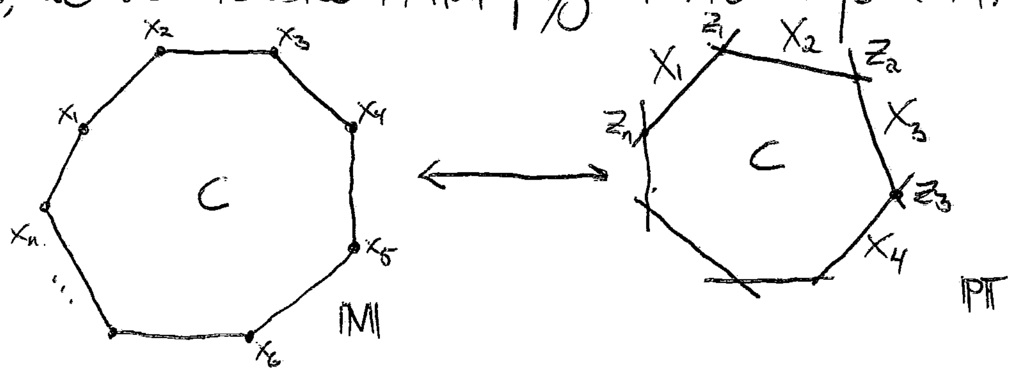
Define $U_X(\lambda,\lambda') \equiv \gamma(x,\lambda)\gamma^{-1}(x,\lambda')$. Then $U_X(\lambda,\lambda') : E|_{X,\lambda'} \to E|_{X,\lambda}$ and $U_X(\lambda,\lambda) = \mathbb{1}$. So this object acts as a parallel propagator along X and can be used to compare different fibers!

Hence, take $\mathcal{O}^G_{(x,\theta)} = \displaystyle\int_{X\times X} \langle\lambda d\lambda\rangle\langle\lambda' d\lambda'\rangle \, \mathrm{tr}\left[U_X(\lambda,\lambda') \frac{\partial^2 \mathcal{A}(\lambda')}{\partial x^a \partial x^b} U_X(\lambda',\lambda) \frac{\partial^2 \mathcal{A}(\lambda)}{\partial x^a \partial x^b} \right]$

$X \cong \mathbb{P}^1 \subset \mathbb{PT}$

Wilson Loops

In all of the conjectures, we are interested in null polygonal Wilson loops in $\mathbb{M}$. Using the incidence relations:



~~Thet~~ On space-time $W_R[C] = \mathrm{tr}_R \, \mathrm{Hol}_x[C]$. The holomorphic frame gives us a means to compute the holonomy of the nodal curve in twistor space:

On PT, ~~At~~ $\mathrm{Hol}_{Z_n}[C] = \mathcal{U}_{X_1}(Z_n, Z_1)\,\mathcal{U}_{X_2}(Z_1, Z_2) \cdots \mathcal{U}_{X_n}(Z_{n-1}, Z_n)$, so we define $W[C] = \mathrm{tr}\left[\mathcal{U}(Z_n, Z_1) \cdots \mathcal{U}(Z_{n-1}, Z_n)\right]$, with R = fundamental representation.

Claim: $W[C]$ agrees with the space-time SUSY Wilson loop for $N=4$ SYM, at least up to terms proportional to the equations of motion.

Using the propagator Δ_* we defined earlier, one can compute the integrands of $\langle W[C]\rangle$ with respect to the twistor action: $\langle W[C]\rangle = \int [DA]\, W[C]\, e^{-S[A]}$ via Wick's theorem.

~~All examples~~

Amplitudes / Wilson loops

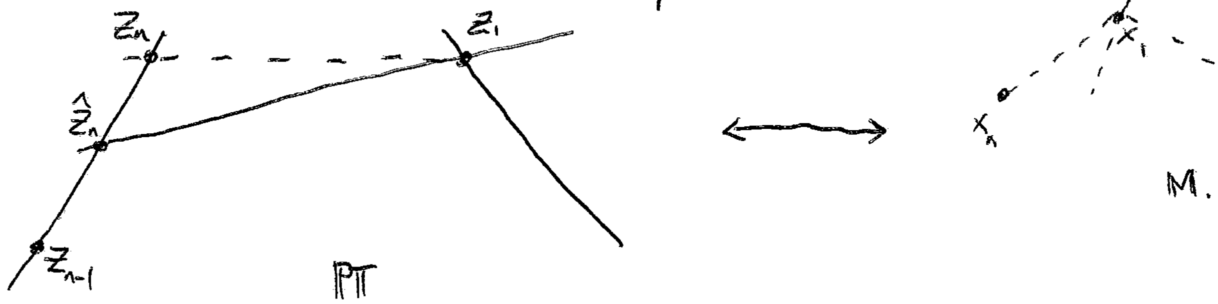
Let $W_k^\ell = W_k^\ell(Z_1, \ldots Z_n; Y_1, \ldots, Y_\ell)$ be the ℓ-loop integrand with Grassman degree $4k$.

All examples give $W_k^\ell = A_k^\ell$, but how can we prove that the two integrands are the same for all ℓ, k?

Scattering amplitudes obey the BCFW recursion relation; this basically fixes their pole structure under a 1-parameter deformation of the external particle momenta.

If we can show that the Wilson loop also obeys this relation, then done.

For the twistor Wilson loop, this corresponds to $\hat{Z}_n(t) = Z_n + t Z_{n-1}$,

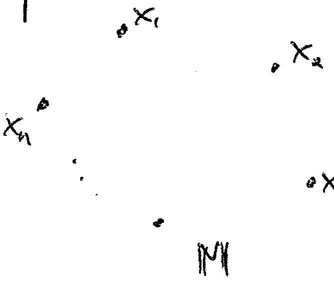


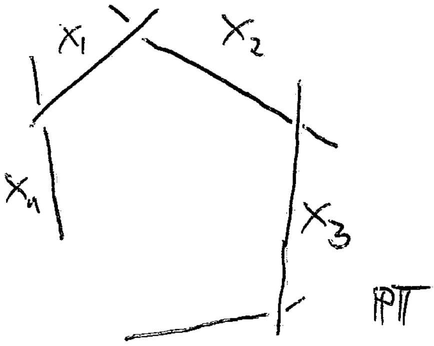
As t varies, the line $(\hat{n}(t), 1)$ sweeps out a plane that will intersect the other components of the Wilson loop, as well as the MHV vertices coming from the perturbative expansion of $I[A]$.

By considering $\bar{\partial}_t \langle W[C_t]\rangle$, it can be shown that these contributions lead to holomorphic loop equations, which reproduce the BCFW recursion relation, completing the proof! [Bullimore + Skinner]

Correlation Functions / Wilson Loops:

$$\lim_{x^2_{i,i+1}\to 0} \frac{\langle O(x_1)\cdots O(x_n)\rangle}{\langle O(x_1)\cdots O(x_n)\rangle^{tree}} :$$





Now, $\langle O(x_1)\cdots O(x_n)\rangle^{tree} \sim \frac{1}{x_{12}^2 x_{23}^2 \cdots x_{n1}^2}$, so unless there are divergences in $\langle O(x_1)\cdots O(x_n)\rangle$ to cancel this, the null limit gives zero!

On twistor space, $\langle O(x_1)\cdots O(x_n)\rangle = \int [DA]\, O(x_1)\cdots O(x_n)\, e^{-S[A]}$. We can perturbatively expand $I[A]$ to give an infinite sum of MHV vertices, and then apply Wick's theorem with respect to $S_{SD}[A]$ to perform the contractions.

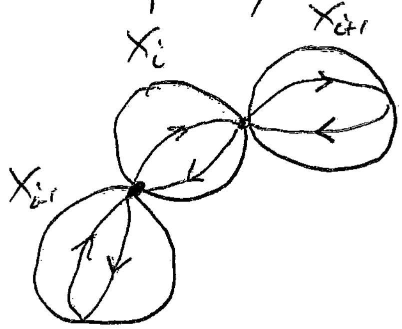
Contractions to consider: - Between an operator and a MHV vertex
- Between operators and frames on the $\{X_i\}\subset \mathbb{PT}$.

All done with the assumptions of normal ordering and genericity (MHV vertices are not null separated from the operator insertions).

Using the twistorial Feynman rules, we can show that all contractions are finite or vanishing in the null limit except:

$$\left\langle \frac{\partial^2 A}{\partial Z^a \partial Z^b}\bigg|_{X_i} \frac{\partial^2 A}{\partial Z^c \partial Z^d}\bigg|_{X_{i+1}} \right\rangle = \frac{\varepsilon_{abcd}}{x^2_{i,i+1}}.$$

Hence, these contractions precisely counter-balance the tree-level denominator, and we are left with



$= \langle W_{adj}[C]\rangle$, at the level of the integrand.

So,
$$\lim_{x^2_{i,i+1}\to 0} \frac{\langle O(x_1)\cdots O(x_n)\rangle}{\langle O(x_1)\cdots O(x_n)\rangle} = \langle W_{adj}[C]\rangle \xrightarrow[\text{limit}]{\text{planar}} \langle W[C]\rangle^2$$

Wilson loops with Local Operators:

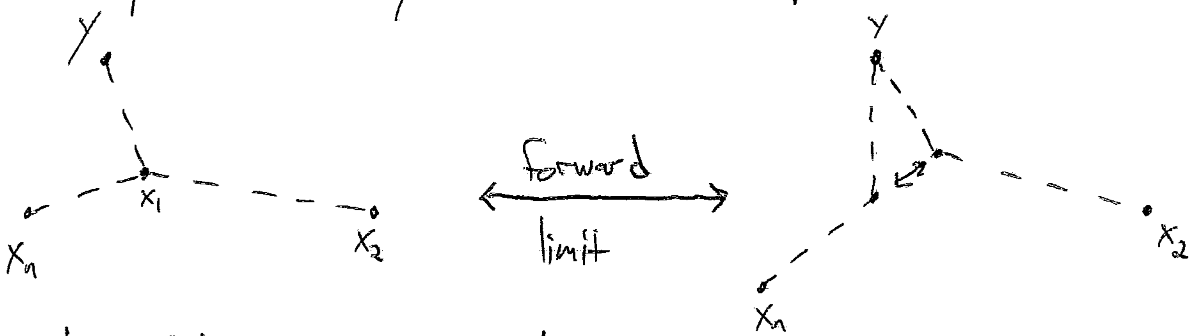
Using the same ~~the~~ techiques, it's easy to investigate the situation where one of the local operators stays in general position.

$$\lim_{x^2_{i,i+1}\to 0} \frac{\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\mathcal{O}(y)\rangle}{\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle} = \lim_{x^2_{i,i+1}\to 0} \frac{\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\mathcal{O}(y)\rangle}{\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle}^{\text{tree}} \frac{1}{\langle W_{adj}[C]\rangle}$$

$$= \frac{\langle W_{adj}[C]\mathcal{O}(y)\rangle}{\langle W_{adj}[C]\rangle} \xrightarrow[\text{limit}]{\text{planar}} 2\,\frac{\langle W[C]\mathcal{O}(y)\rangle}{\langle W[C]\rangle}.$$

We can also derive BCFW-type recursion relations for those mixed correlators using the same techniques as before.

Now there is an additional contribution to $\bar{\partial}_t\langle W[C_t]\mathcal{O}(y)\rangle$ from where x_1 becomes null separated from y; this can be interpreted as a kind of forward limit:



Works out beautifully in twistor space, but may be unique to ½-BPS operators...