Liouville Quantum Gravity, KPZ & Schramm-Loewner Evolution III

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– UK-Japan Winter School 2012 –

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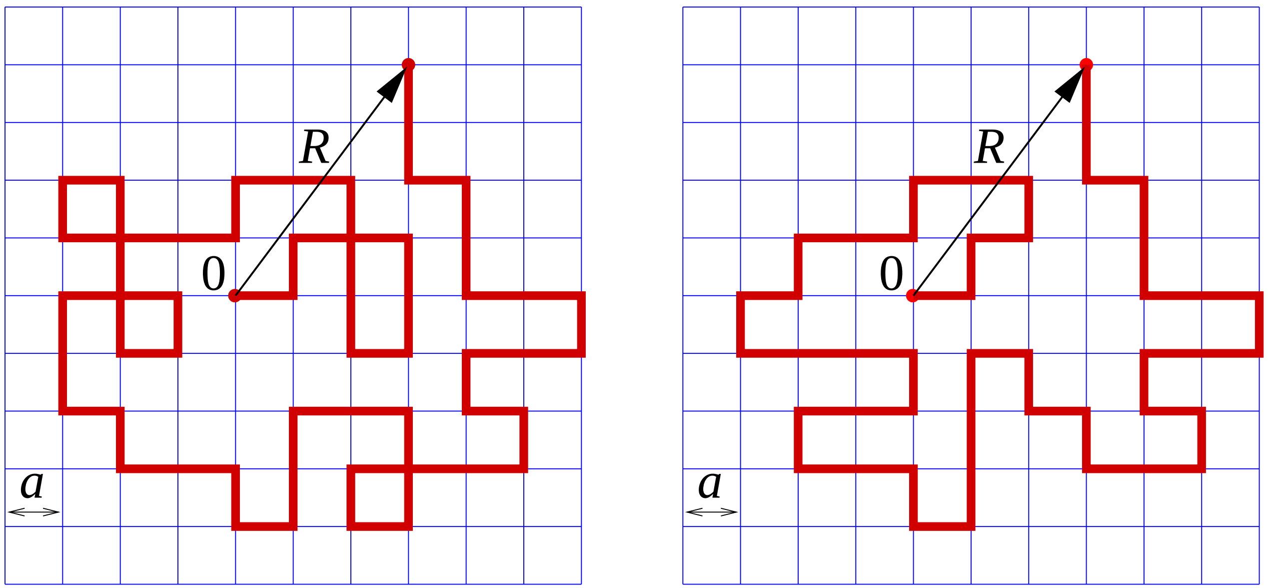
University of Oxford / 5 – 8 January 2012

Liouville Quantum Gravity

& SLE

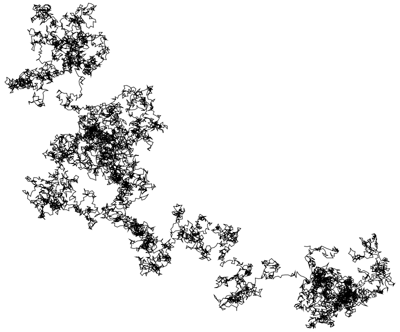
Random Paths

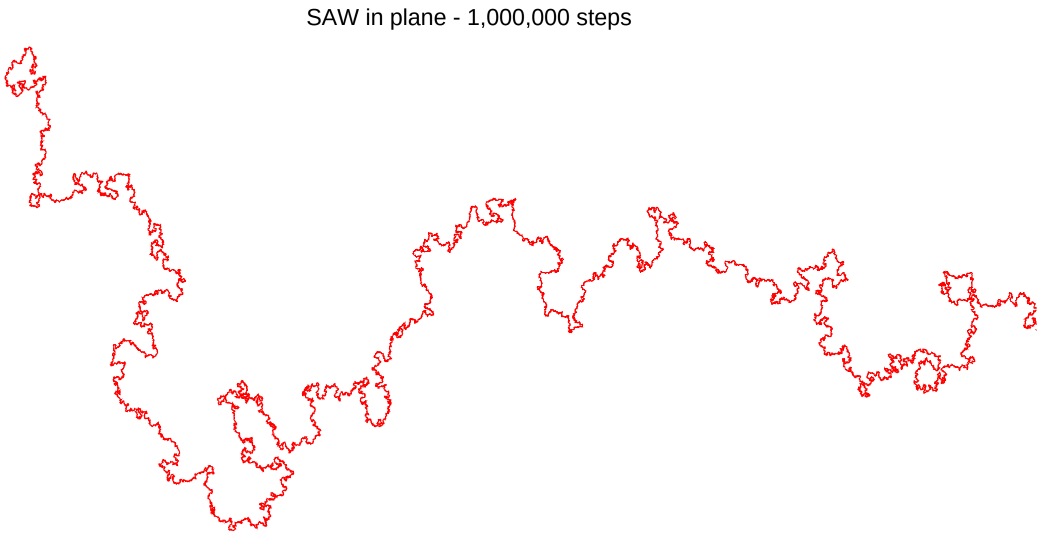


Random walk and *self-avoiding* walk with N steps on the square lattice; mesh size a :

$$\mathbb{E}(R^2)_N = Na^2; \qquad \mathbb{E}(R^2)_N = \bullet N^{2\nu} a^2, \;\; N \to \infty$$

Continuum Limit





From random walk to self-avoiding walk (SLE$_{\kappa=8/3}$):
Complete change of statistical and geometrical properties
(here in two dimensions)

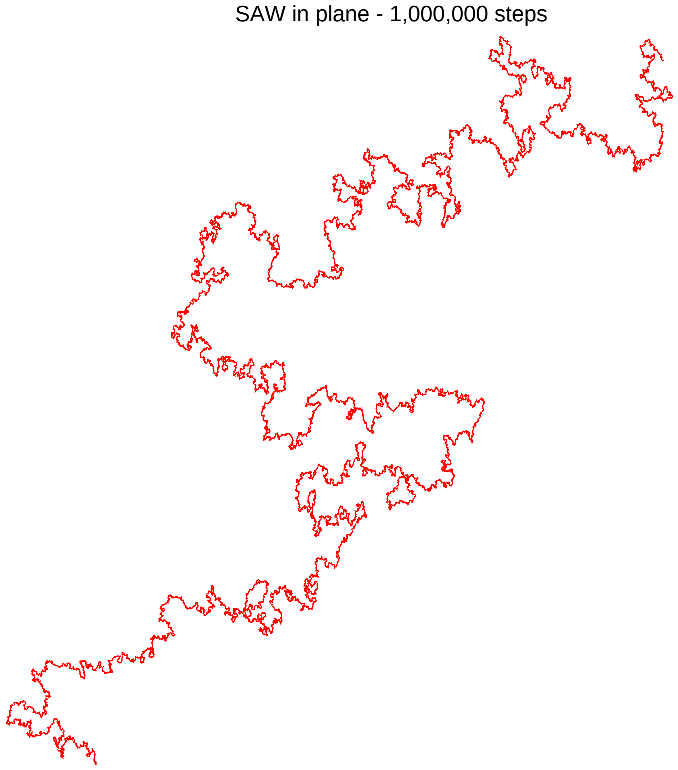
B. Nienhuis '82 : $\mathbb{E}(R^2) \propto N^{3/2}a^2,\ D_{\mathrm{Hausdorff}} = \frac{4}{3}$

Brownian Path

J. Perrin (1909), N. Wiener (1923) scale invariance

P. Lévy (1942) planar conformal invariance

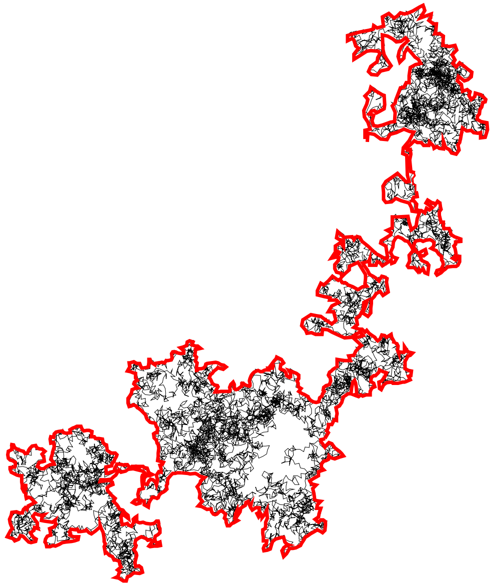
Self-Avoiding Walk & SLE$_{\kappa=8/3}$



(Courtesy of T. Kennedy)

$$D_{\text{Hausdorff}} = \frac{4}{3}$$

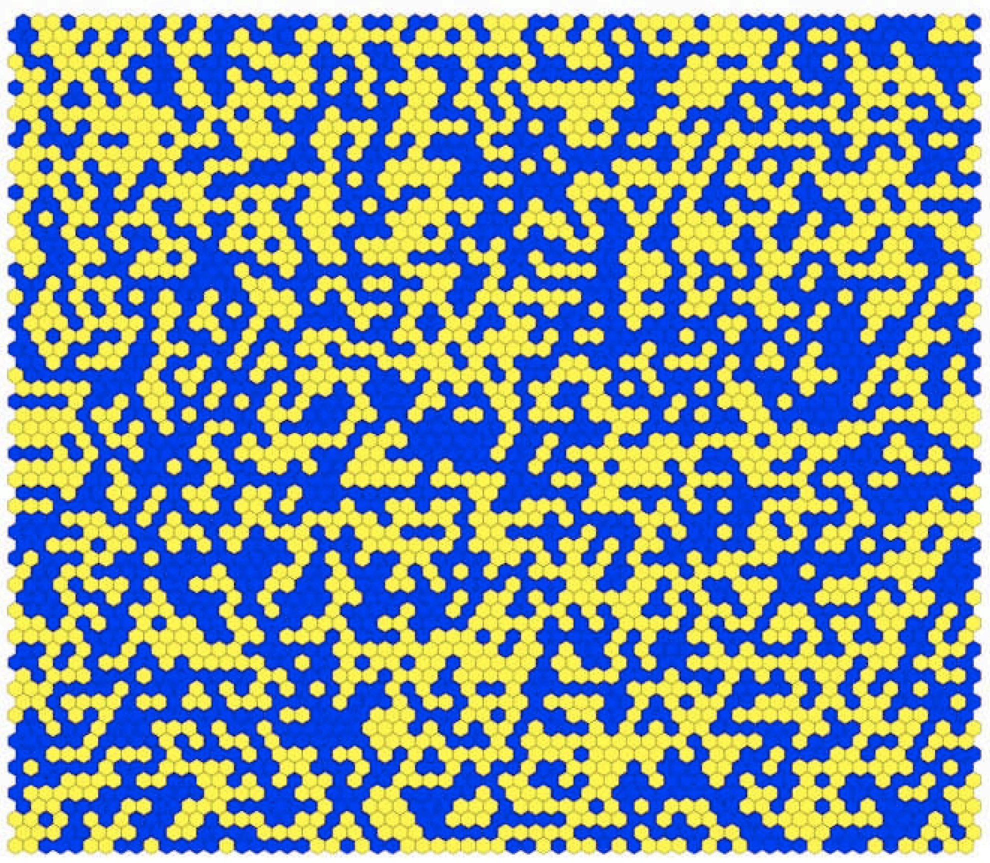
Planar Brownian Frontier



- *Mandelbrot conjecture '82:* $D_{\text{Hausdorff}} = 4/3$
- *The Brownian frontier is the scaling limit of a self-avoiding walk* [$\text{SLE}_{\kappa=8/3}$]

G.-F. Lawler, O. Schramm, W. Werner, '01 (2006 Fields Medal)

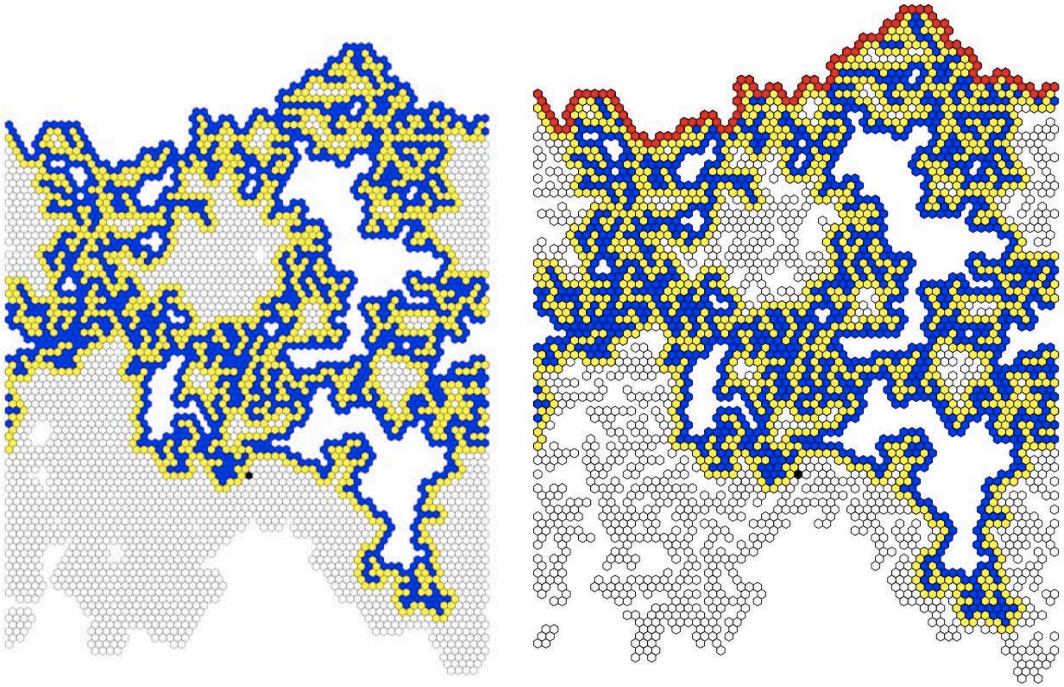
Percolation $(p_c = 1/2)$



(Courtesy of R. Ziff, U. Michigan)

M. Aizenman; J. Cardy, '92; S. Smirnov, '01 (2010 Fields Medal)

Percolation Cluster Hull & Frontier

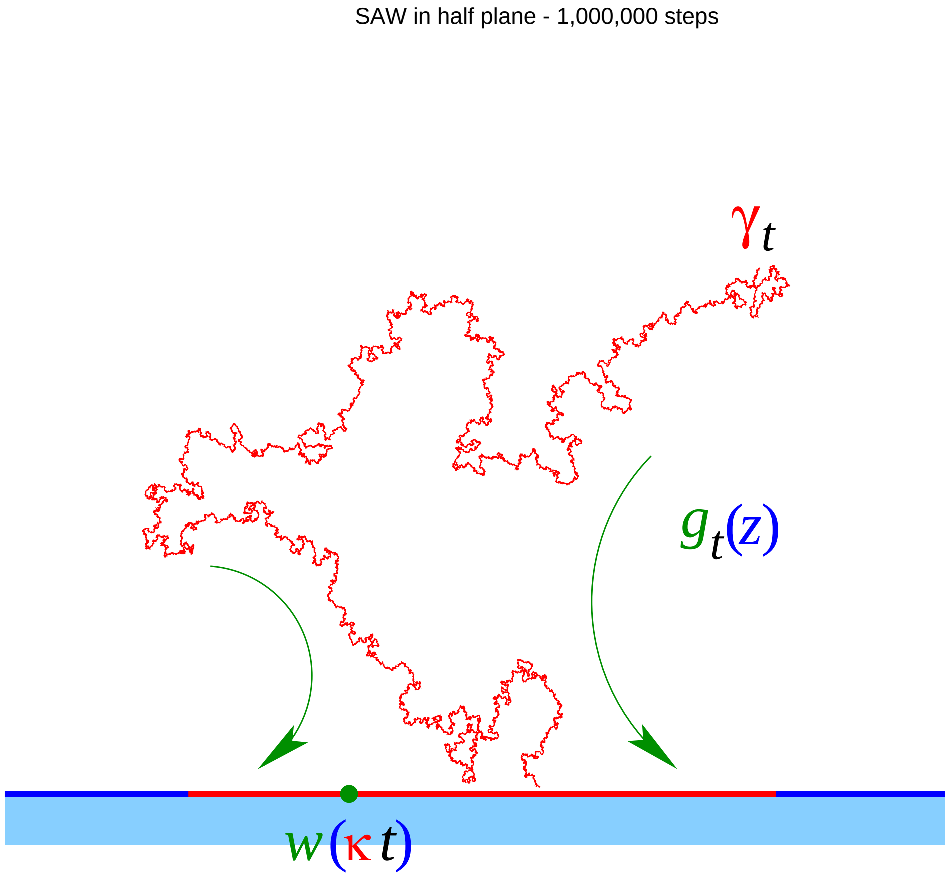


(Courtesy of R. Ziff, U. Michigan)

Duality • *Hull:* $D_{\mathrm{H}} = \frac{7}{4}$ $(\mathrm{SLE}_{\kappa=6})$

• *External Perimeter:* $D_{\mathrm{EP}} = \frac{4}{3}$ $(\mathrm{SLE}_{\kappa=8/3})$

Schramm-Loewner Evolution (SLE$_\kappa$, 2000)



$$\partial_t g_t(z) = 2/[g_t(z) - \sqrt{\kappa} B_t]$$

Simple path for $\kappa \in [0,4]$, *self-bouncing for* $\kappa \in (4,8)$, $\kappa \geqslant 8$ *space filling*

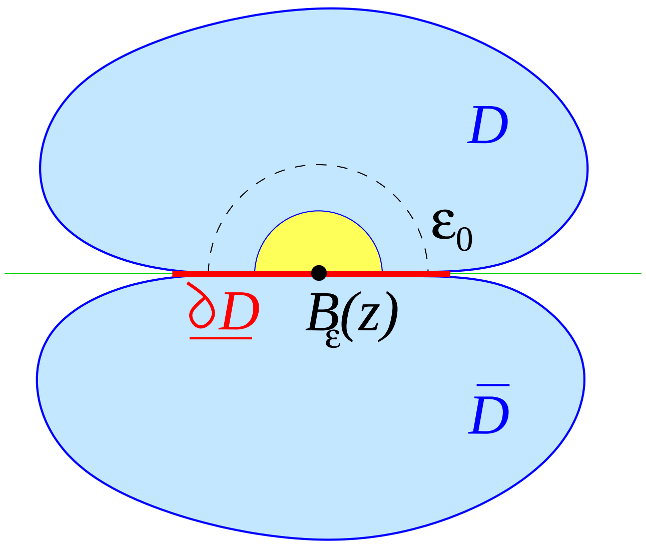
SLE - GFF (QG) Coupling

(Dubédat, 2009)

Sheffield, arXiv:1012.4797

D. & Sheffield, PRL **107***, 131305 (2011), arXiv:1012.4800*

Bulk & Boundary Liouville Quantum Gravity



- *GFF with free boundary conditions on* $\underline{\partial D}$
- *Circle averages* $h_\varepsilon(z)$, $z \in D$
- *Half-circle averages* $\hat{h}_\varepsilon(z)$, $z \in \underline{\partial D}$.

Quantum Area Measure

$$d\mu_\varepsilon := \exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2} d^2 z$$

converges, as $\varepsilon \to 0$ *and for* $\gamma < 2$, *to a random measure, denoted by* $e^{\gamma h(z)} d^2 z$.

Quantum Boundary Measure

$$d\hat{\mu}_\varepsilon := \exp\left[\frac{\gamma}{2} \hat{h}_\varepsilon(z)\right] \varepsilon^{\gamma^2/4} dz$$

converges, as $\varepsilon \to 0$ *and for* $\gamma < 2$, *to a boundary random measure, denoted by* $e^{\gamma h(z)} dz$.

Quantum fractal measures and KPZ.—We discuss now Euclidean and quantum *fractal measures*. The d-dimensional *Euclidean* or analogously *quantum measure* of planar *fractal* sets is characterized by scaling properties:

- If we rescale a d-dimensional fractal $X \subset \mathcal{D} \subset \mathbb{C}$ via the map $z \to \psi(z) = bz$, $b \in \mathbb{C}$ (so that the Euclidean area of the domain $\mathcal{D}$ is multiplied by $|b|^2$) then the *d-dimensional Euclidean fractal measure* of X is multiplied by $|b|^d = |b|^{2-2x}$, where x (the *Euclidean scaling weight*) is defined by $d := 2 - 2x (\leqslant 2)$.
- If X is a fractal subset of a random surface $\mathcal{S} := (\mathcal{D}, h)$, and we rescale $\mathcal{S}$ so that its quantum area increases by a factor of $|b|^2$, then the *quantum fractal measure* $\mathcal{Q}(X, h)$ of X is multiplied by $|b|^{2-2\Delta}$, where Δ is the analogous *quantum scaling weight*.

- $\mathcal{Q}\left(\psi(X,h)\right) = \mathcal{Q}(X,h)$ whenever ψ is conformal and

$$\begin{aligned} \psi(\mathcal{D},h) &:= \left(\psi(\mathcal{D}), h\circ\psi^{-1} - Q\log|\psi'|\right) && (1)\\ Q &:= \frac{\gamma}{2}+\frac{2}{\gamma}. && (2) \end{aligned}$$

This is because the pair $\mathcal{S} = (\mathcal{D},h)$ describes the same Liouville quantum surface (up to coordinate change) as the conformally transformed pair $\psi(\mathcal{D},h)$.

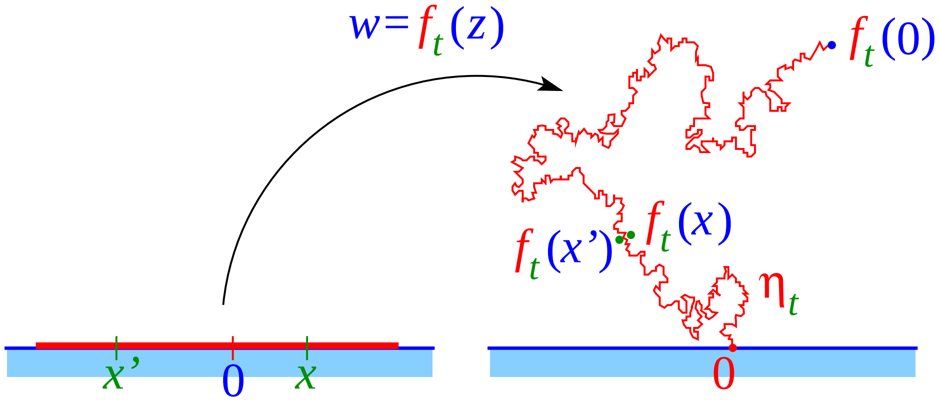
- Then from (2) the relation

$$d = \alpha Q - \alpha^2/2,$$

where $d := 2-2x$ and $\alpha := \gamma(1-\Delta)$, is equivalent to KPZ

$$x = (\gamma^2/4)\Delta^2 + \left(1-\gamma^2/4\right)\Delta \quad \square$$

"Zipping-up" SLE Map



Let f_t be the (reverse) SLE$_\kappa$ conformal map

$$z \in \mathbb{H} \to w = f_t(z) \in \mathbb{H} \setminus \eta_t,$$

with trace η_t and tip $f_t(0)$ [$t = 0$, $f_0(z) = z$]. It satisfies the stochastic differential equation (B_t *standard Brownian motion*)

$$d\, f_t(z) = -2dt / f_t(z) - \sqrt{\kappa}\, dB_t.$$

(Reverse) SLE Martingale

Real stochastic process in the upper-half plane:

$$\begin{aligned} \mathfrak{h}_0(z) &:= \frac{2}{\sqrt{\kappa}} \log|z|, \\ \mathfrak{h}_t(z) &:= \mathfrak{h}_0 \circ f_t(z) + Q \log|f_t'(z)|. \end{aligned}$$

This process $\mathfrak{h}_t(z)$ is a *martingale* (so that $\mathbb{E}\mathfrak{h}_t(z) = \mathfrak{h}_0(z)$) for the particular choice:

$$Q = \sqrt{\kappa}/2 + 2/\sqrt{\kappa},$$

for which $d\mathfrak{h}_t(z) = -\Re[2/f_t(z)]dB_t$.

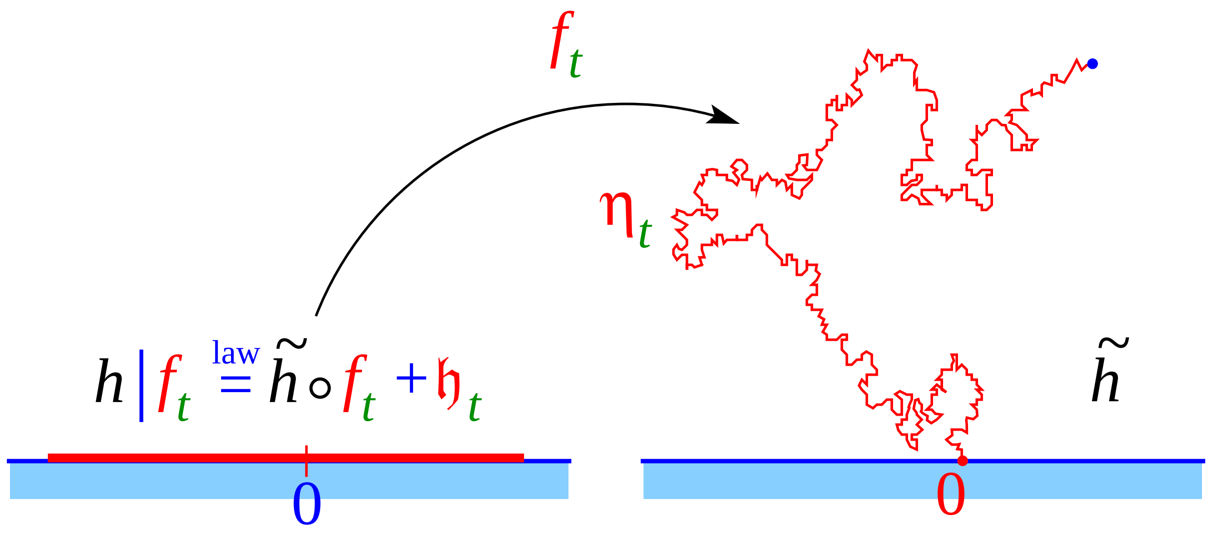
Itô Calculus & Martingales

Real stochastic process for $t \geqslant 0$ and $z \in \mathbb{H}$:

$$\begin{aligned} \mathfrak{h}_0(z) &:= \frac{2}{\sqrt{\kappa}} \log|z|, \\ \mathfrak{h}_t(z) &:= \frac{2}{\sqrt{\kappa}} \log|f_t(z)| + Q \log|f_t'(z)|. \end{aligned} \tag{3}$$

By stochastic *Itô calculus* (i.e., using the *Brownian local covariations* $d\langle B_t, B_t\rangle = (dB_t)^2 = dt$, $d\langle B_t, t\rangle = dB_t dt = 0$ and $d\langle t, t\rangle = (dt)^2 = 0$), the particular choice in (3), $Q = \sqrt{\kappa}/2 + 2/\sqrt{\kappa}$, gives a *driftless diffusion process* $d\mathfrak{h}_t(z) = -R_t(z)dB_t$, with $R_t(z) := \Re[2/f_t(z)]$. Then $\mathfrak{h}_t(z)$ is a time-changed Brownian motion (called a *local martingale*) with *local covariation* $d\langle \mathfrak{h}_t(y), \mathfrak{h}_t(z)\rangle = R_t(y)R_t(z)dt$, having the further *martingale* property $\mathbb{E}\mathfrak{h}_t(z) = \mathfrak{h}_0(z)$.

SLE–GFF Coupling



Define $h := \tilde{h} + \mathfrak{h}_0$, sum of the GFF $\tilde{h}$ on $\mathbb{H}$ with *free boundary conditions* on $\mathbb{R}$, and of the deterministic function $\mathfrak{h}_0$. Given f_t, the conditional law of h (denoted by $h|f_t$) is

$$h(z)|f_t \overset{\text{(law)}}{=} \tilde{h}\circ f_t(z) + \mathfrak{h}_t(z),$$

where $\tilde{h}\circ f_t$ is the pullback of the free boundary GFF $\tilde{h}$.

SLE–GFF Coupling

This h can be coupled *[Sheffield, 2010]* with the reverse Loewner evolution f_t so that, *given* f_t, the conditional law of h (denoted by $h| f_t$) is

$$h(z)|\, f_t \overset{\text{(law)}}{=} \tilde{h} \circ f_t(z) + \mathfrak{h}_t(z), \tag{4}$$

where $\tilde{h} \circ f_t$ is the pullback of the free boundary GFF $\tilde{h}$ in the image half-plane, and where $\mathfrak{h}_t$ is the martingale (3). This means that to sample h, one can first sample the B_t process (which determines f_t), then sample independently the f.b.c. GFF $\tilde{h}$ and take (4).
The conditional expectation of (4) w.r.t. $\tilde{h}$ is the *martingale* $\mathbb{E}\left[h(z)|\, f_t\right] = \mathfrak{h}_t(z)$.

Neumann Green function

Consider the *Neumann Green function* in $\mathbb{H}$, $G_0(y,z) := -\log(|y-z||y-\bar{z}|)$, and define the *time-dependent* $G_t(y,z) := G_0\big(f_t(y), f_t(z)\big)$, i.e., G_0 taken at image points under f_t. A simple calculation of the Green function's variation shows that $-dG_t(y,z) = d\langle \mathfrak{h}_t(y), \mathfrak{h}_t(z)\rangle$ (*Hadamard's formula*). Integrating w.r.t. t yields the covariation of the $\mathfrak{h}_t$ martingales

$$\langle \mathfrak{h}_t(y), \mathfrak{h}_t(z)\rangle = G_0(y,z) - G_t(y,z).$$

Taking the limit $y \to z$ in the latter, one obtains

$$\langle \mathfrak{h}_t(z), \mathfrak{h}_t(z)\rangle = C_0(z) - C_t(z),$$

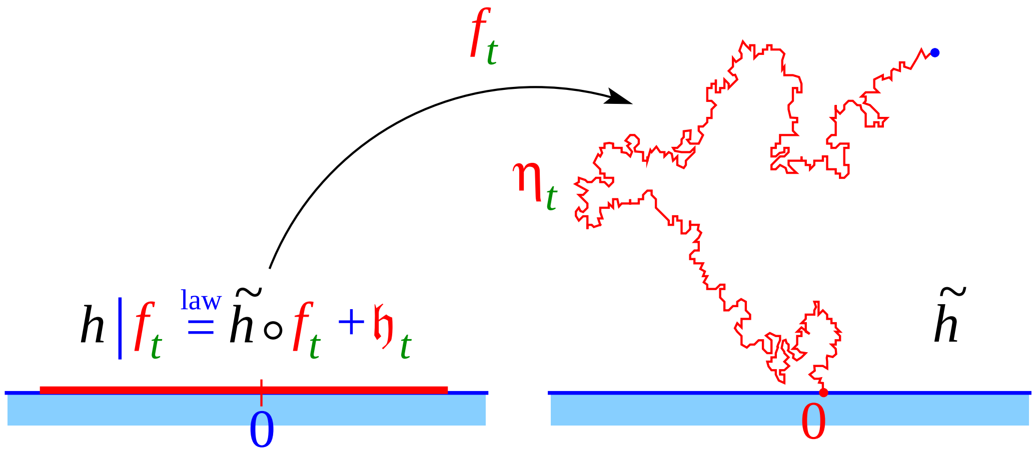
where $C_t(z) := -\log\left[\Im f_t(z) |f_t'(z)|\right]$.

SLE–GFF Coupling

Define the *covariance*: $\mathrm{Cov}[A,B] := \mathbb{E}[AB] - \mathbb{E}[A]\mathbb{E}[B]$.
Recall that the Green's function $G_0(y,z) = \mathrm{Cov}[\tilde{h}(y), \tilde{h}(z)]$, thus $G_t(y,z) = \mathrm{Cov}[\tilde{h} \circ f_t(y), \tilde{h} \circ f_t(z)]$. *The random distribution $\tilde{h} \circ f_t$ and the set of (time changed) Brownian motions $\mathfrak{h}_t$ are Gaussian processes, whose respective covariance G_t and covariation $\langle \mathfrak{h}_t, \mathfrak{h}_t \rangle$ thus add to constant G_0:*

$$\begin{aligned} G_t(y,z) + \langle \mathfrak{h}_t(y), \mathfrak{h}_t(z) \rangle &= G_0(y,z) \\ \mathrm{Cov}[\tilde{h} \circ f_t(y), \tilde{h} \circ f_t(z)] + \langle \mathfrak{h}_t(y), \mathfrak{h}_t(z) \rangle &= \mathrm{Cov}[\tilde{h}(y), \tilde{h}(z)] \\ &= \mathrm{Cov}[h(y), h(z)] \quad \square \end{aligned}$$

Liouville Invariance



Since $\mathfrak{h}_t := \mathfrak{h}_0 \circ f_t + Q \log |f_t'|$, and $h := \tilde{h} + \mathfrak{h}_0$, we get $\tilde{h} \circ f_t + \mathfrak{h}_t = h \circ f_t + Q \log |f_t'|$. For Q given by (2), $Q = \gamma/2 + 2/\gamma$, this is precisely the transformation law (1) of the GFF h under the conformal map f_t^{-1}. Then the pair $(\mathbb{H}, \tilde{h} \circ f_t + \mathfrak{h}_t) = f_t^{-1}(\mathbb{H} \setminus \eta_t, h)$ describes the same random surface as the pair $(\mathbb{H} \setminus \eta_t, h)$: *Given f_t, the image under f_t of the measure $e^{\gamma h(z)} d^2 z$ in $\mathbb{H}$ is a random measure whose law is the a priori (unconditioned) law of $e^{\gamma h(w)} d^2 w$ in $\mathbb{H} \setminus \eta_t$.*

Liouville Quantum Measure

$$\left(e^{\gamma h(z)} \,|\, f_t\right) d^2 z \overset{\text{(law)}}{=} e^{\gamma h(w)}\, d^2 w \quad \text{(conformal invariance)}$$

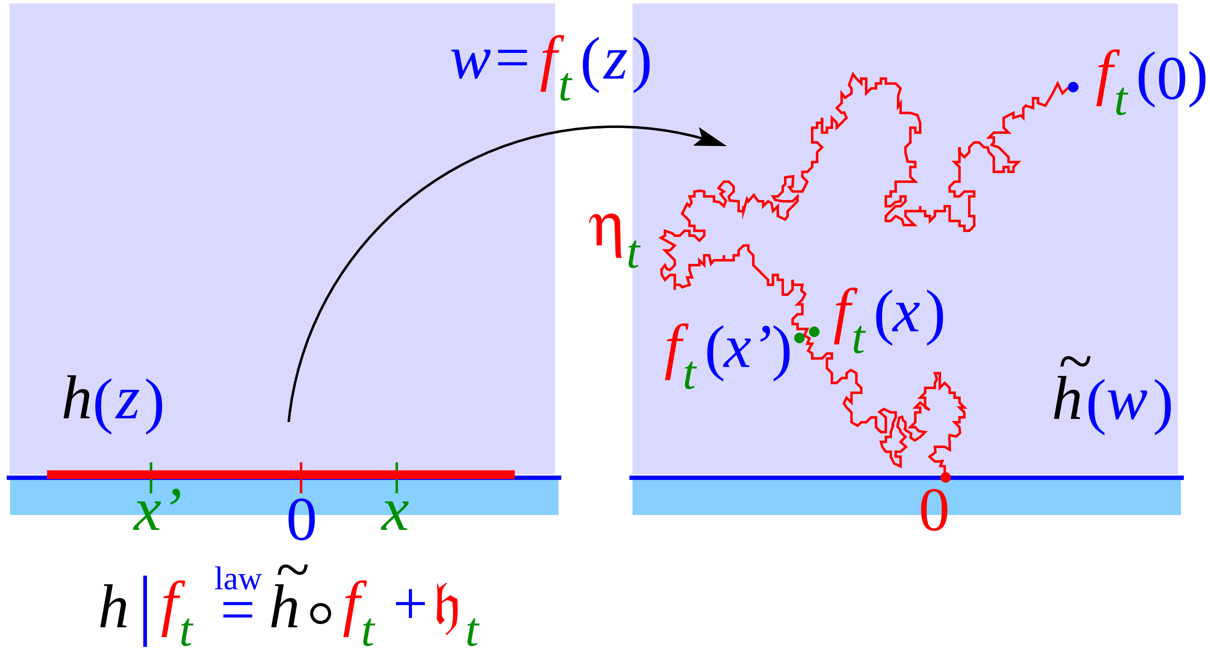
for $d = 2 = \gamma Q - \gamma^2/2$, *i.e.*, $Q = \gamma/2 + 2/\gamma = \sqrt{\kappa}/2 + 2/\sqrt{\kappa}$

$$\Longrightarrow \ \gamma = \sqrt{\kappa} \wedge \left(4/\sqrt{\kappa}\right), \ \ \gamma' = 4/\gamma$$

- $\gamma \leqslant 2$: *KPZ prediction* $\gamma = \left(\sqrt{25-c} - \sqrt{1-c}\right)/\sqrt{6}$ for the *central charge* $c = \frac{1}{4}(6-\kappa)(6-16/\kappa) \leqslant 1$ of the SLE's CFT coupled to gravity.
- $\gamma' = 4/\gamma > 2$: *Duality* property of Liouville quantum gravity; the quantum measure develops atoms with localized area.

Conformally welding two γ-Liouville quantum surfaces produces SLE$_\kappa$.

Conformal Welding



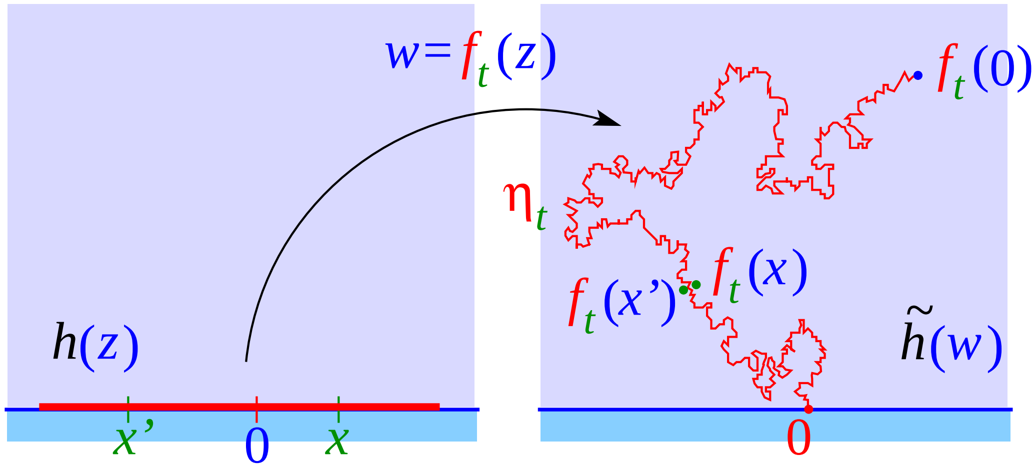
Conformal welding: the *quantum boundary lengths* of any pair of real segments $[0, x]$ and $[x', 0]$ such that $f_t(x) = f_t(x')$ on the SLE trace are *a.s. equal* for $h = \tilde{h} + \mathfrak{h}_0$ *[Sheffield, 2010].*

Expected Liouville Quantum Area

For $\gamma = \sqrt{\kappa \wedge 16/\kappa}$

$$
\begin{aligned}
d\mathcal{A} \;&:=\; d^2z\,\mathbb{E}[e^{\gamma h(z)} \,|\, f_t] = d^2w\,\mathbb{E}[e^{\gamma h(w)}] \\
&=\; d^2w\,|w|^{2-\kappa/2}(\sin\varphi)^{-\kappa/2}, \kappa \leqslant 4 \\
&=\; d^2w(\sin\varphi)^{-8/\kappa}, \kappa \geqslant 4 \\
&\quad\; (\varphi := \arg w)
\end{aligned}
$$

Liouville Quantum Gravity & SLE



$$h \,|\, f_t \overset{\text{law}}{=} \tilde{h} \circ f_t + \mathfrak{h}_t$$

- *Conformally welding two γ-Liouville quantum boundaries yields SLE$_\kappa$ for*

$$\gamma = \sqrt{\kappa} \wedge (4/\sqrt{\kappa}) = \left(\sqrt{25-c} - \sqrt{1-c}\right)/\sqrt{6} < 2 \quad \text{(KPZ II)}$$

- *Exponential martingales yield SLE quantum measures:*

$$\mathbb{E}\left[h | f_t\right] = \mathfrak{h}_t, \quad \mathbb{E}\left(e^{\alpha h} | f_t\right) = \exp\left[\alpha \mathfrak{h}_t - (\alpha^2/2)\langle \mathfrak{h}_t, \mathfrak{h}_t \rangle\right]$$

[D. & Sheffield, PRL **107***, 131305 (2011)]*

SLE Exponential Martingales & KPZ

$$\begin{aligned}
\mathcal{M}_t^{\alpha}(z) &:= \mathbb{E}\left(e^{\alpha h(z)} \,|\, f_t\right), \;\; \alpha \in \mathbb{R} \\
\left(e^{\alpha h(z)} \,|\, f_t\right) d^2 z &\overset{(\text{law})}{=} \left|f_t'(z)\right|^{d-2} e^{\alpha h(w)} d^2 w \\
d &:= \alpha Q - \alpha^2/2 \;\; (\text{KPZ})
\end{aligned}$$

where $w = f_t(z)$, $d^2 w = |f_t'(z)|^2 d^2 z$.

SLE Exponential Martingales

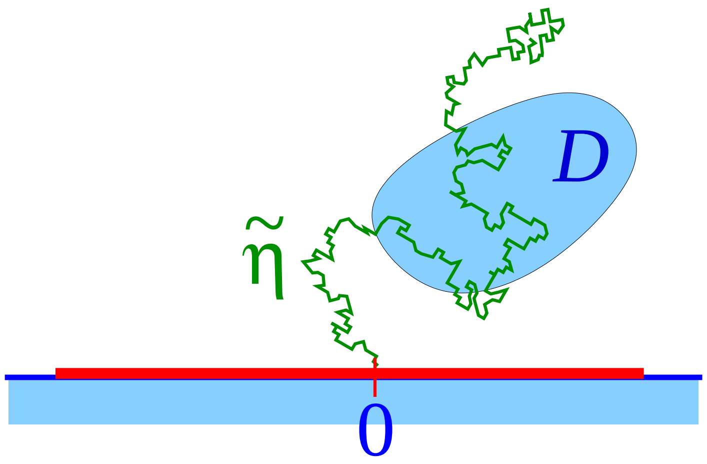
- Conditional expectation w.r.t. GFF h: $\mathbb{E}\left[h(z) \,|\, f_t\right] = \mathfrak{h}_t(z)$.
- Conditional expectations of exponentials:

$$\begin{aligned}
\mathcal{M}_t^{\alpha}(z) &:= \mathbb{E}\left(e^{\alpha h(z)} \,|\, f_t\right), \;\; \alpha \in \mathbb{R} \\
&= \exp\left[\alpha \mathfrak{h}_t(z) - (\alpha^2/2) C_t(z)\right] \\
&= \left|f_t'(z)\right|^d |w|^{2\alpha/\sqrt{\kappa}} (\Im w)^{-\alpha^2/2}; \; d := \alpha Q - \alpha^2/2 \\
C_t(z) &:= \langle \mathfrak{h}_t(z), \mathfrak{h}_t(z) \rangle = \log[\Im f_t(z) \,|\, f_t'(z)|]
\end{aligned}$$

where $w = f_t(z)$; $\mathcal{M}_t^{\alpha}(z)$ *is an exponential martingale with respect to the Brownian motion driving the SLE process:*

$$\mathbb{E}\mathcal{M}_t^{\alpha}(z) = \mathcal{M}_0^{\alpha}(z) = |z|^{2\alpha/\sqrt{\kappa}} (\Im z)^{-\alpha^2/2}.$$

SLE Natural Length



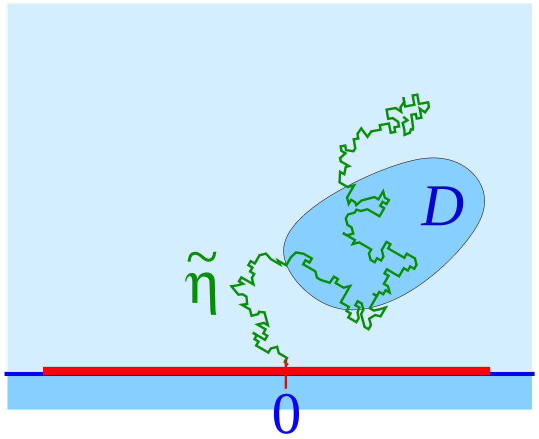
Expected (w.r.t. the $\text{SLE}_{\kappa\in[0,8]}$ law) *length* of an infinite SLE $\tilde{\eta}$ in D *(Lawler & Sheffield, 2009)*

$$\nu(D) = \int_D G(z)d^2z,$$

SLE Green's function in $\mathbb{H}$:

$$G(z) := |z|^a |\Im z|^b, \ a = 1 - 8/\kappa, \ b = 8/\kappa + \kappa/8 - 2.$$

SLE Quantum Length



$$h = \tilde{h} + \mathfrak{h}_0$$

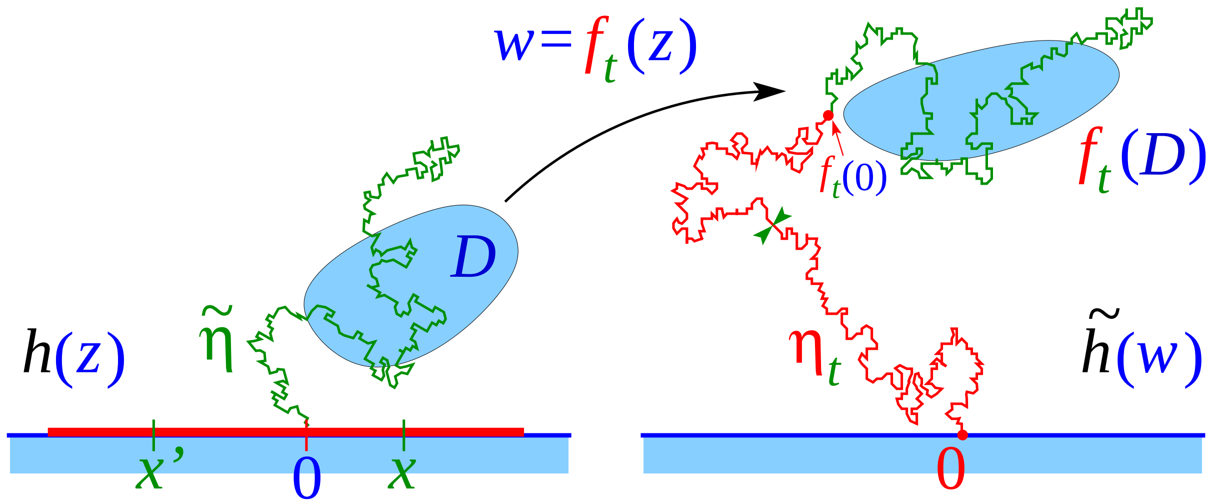
Expected (w.r.t. $\tilde{\eta}$, given h) Liouville *quantum length* ν_Q in D

$$\nu_Q(D,h) := \int_D e^{\alpha h(z)} G(z) d^2 z,$$

$\alpha = \sqrt{\kappa}/2$ ($= \gamma/2$ for $\kappa \leqslant 4$, and $\gamma'/2$ for $\kappa > 4$) satisfies KPZ for the SLE Hausdorff dimension $d = 1 + \kappa/8$.

[Doob Meyer, second moment method.]

Expected SLE Quantum Length



$$h\,|\,f_t \overset{\text{law}}{=} \tilde{h}\circ f_t + \mathfrak{h}_t$$

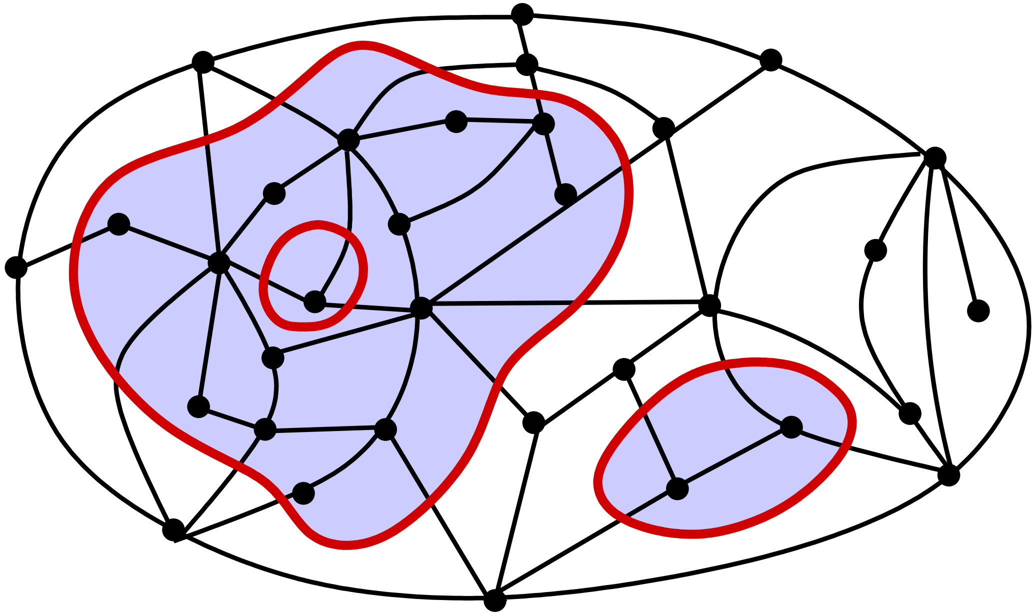
$$\begin{aligned}\mathbb{E}[\nu_Q(D,h)\,|\,f_t] &= \int_D \mathcal{M}_t^{\alpha}(z)G(z)d^2z \\ \mathbb{E}\nu_Q(D,h) &= \int_D \mathcal{M}_0^{\alpha}(z)G(z)d^2z = \int_D (\sin\vartheta)^{8/\kappa-2}d^2z,\end{aligned}$$

with $\vartheta := \arg z$. It is finite for $\kappa \in [0,8)$ and coincides with the *Euclidean area* of D for $\kappa = 4$.

Perspectives

- *Scaling limits of discrete models on random planar graphs*
- *Quantum wedges and cones*
- *Quantum bubbles and foam* $(\gamma\gamma' = 4$ *duality*$)$
- *Geodesics & random metrics*

$\diamond \quad \diamond \quad \diamond$



Quadrangulation with a loop model (courtesy of E. Guitter).